Modal Algebra of Multirelations

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Abstract

We formalise the modal operators from the concurrent dynamic logics of Peleg, Nerode and Wijesekera in a multirelational algebraic language based on relation algebras and power allegories, using relational approximation operators on multirelations developed in a companion article. We relate Nerode and Wijesekera's box operator with a relational approximation operator for multirelations and two related operators that approximate multirelations by different kinds of deterministic multirelations. We provide an algebraic soundness proof of Goldblatt's axioms for concurrent dynamic logic and one for a multirelational Hoare logic based on Nerode and Wijesekera's box as applications.

1 Introduction

This is the third article in a trilogy on the inner structure of multirelations [FGS23b], the determinisation of such relations [FGS23a] and their algebras of modal operators.

Multirelations are relations of type $X \nrightarrow \mathcal{P}Y$, which model alternating nondeterminism. We contribute to a line of work on modal algebras of multirelations [FS15, FS16] related to Peleg's concurrent dynamic logic [Pel87], which has a multirelational semantics, and on algebraic languages for multirelations [FKST17]. These languages are extensions of relation algebras [Sch11] and boolean power allegories [FŠ90] with specific operations for multirelations.

Our main motivation has been the algebraic formalisation of Nerode and Wijesekera's modal box operator [NW90] for concurrent dynamic logic. In dynamic logics, relational modal box operators

$$[S]_r Q = \{a \in X \mid \forall b \in Y.\ S_{a,b} \Rightarrow b \in Q\}$$

for the relation $S : X \nrightarrow Y$ and the set $Q \subseteq Y$ typically express correctness specifications of relational programs: $[S]_r Q$ determines the set of states from which every terminating execution of program S must be in the set Q. Yet Peleg's multirelational box operator

$$[R]_* Q = \{a \mid \forall B.\ R_{a,B} \Rightarrow B \cap Q \neq \emptyset\}$$

fails to capture the correctness of the multirelational program $R : X \nrightarrow \mathcal{P}Y$ in the presence of non-terminating elements of the form $(a, \emptyset)$ [NW90]. Nerode and Wijesekera therefore propose an alternative multirelational box

$$[R]_\alpha Q = \{a \mid \forall B.\ R_{a,B} \Rightarrow B \subseteq Q\}.$$

Goldblatt has subsequently reduced it to a relational box, $[R]_\alpha Q = [\alpha(R)]_r Q$, approximating the multirelation $R : X \nrightarrow \mathcal{P}Y$ by the relation $\alpha(R) : X \nrightarrow Y$ given by

$$\alpha(R) = \{(a, b) \mid b \in \bigcup\{B \mid (a, B) \in R\}\},$$

in which the alternating structure of R has been atomised. But how can $[-]_\alpha$ and $\alpha(-)$ be formalised in an algebraic multirelational language? Which operations and constructs are needed for expressing it and for studying its relationship to $[-]_*$?

A key observation is that Goldblatt's map $\alpha : (X \nrightarrow \mathcal{P}Y) \to (X \nrightarrow Y)$ can be expressed using fundamental concepts of power allegories. For any multirelation $R : X \to \mathcal{P}Y$,

$$\alpha(R) = R\ni,$$

the relational composition of R with the converse of the element-(multi)relation $\in : Y \nrightarrow \mathcal{P}Y$. Likewise we can define a new De Morgan dual diamond operator $\langle - \rangle_\alpha = \langle - \rangle_r \circ \alpha$. Yet the combination of $[-]_\alpha$ with Peleg's multirelational diamond $\langle - \rangle_*$, which is used in Wijesekera and Nerode's as well as in Goldblatt's concurrent dynamic logics, and the study of the relationships between the different modal operators requires further multirelational operations, including the Peleg composition of multirelations. Due to the complexity of their interactions we consider them in concrete extensions and enrichments of the category **Rel**, but with a view to future axiomatic approaches, and therefore by and large through algebraic proofs.

Modal operators on relations or multirelations usually map relations $X \nrightarrow Y$ or multirelations $X \nrightarrow \mathcal{P}Y$ to functions or "predicate transformers" $\mathcal{P}Y \to \mathcal{P}X$. Modal diamonds then arise as relational or multirelational preimage operations and modal boxes as their De Morgan duals. Transformers $\mathcal{P}X \to \mathcal{P}Y$ are obtained by opposition. These preimages can be expressed algebraically using relational and multirelational domain operations together with relational or Peleg composition. Alternatively, via the isomorphism between relations $X \nrightarrow Y$ and functions $X \to \mathcal{P}Y$, predicate transformers $\mathcal{P}X \to \mathcal{P}Y$ can be obtained as Kleisli extensions of Kleisli arrows in the powerset monad on **Set**. Similarly, we can map multirelations $X \nrightarrow \mathcal{P}Y$ to "relational predicate transformers" $\mathcal{P}X \nrightarrow \mathcal{P}Y$, using a Kleisli lifting for multirelations introduced in [FKST17], while relations $X \nrightarrow Y$ are sent to such transformers by the relational image functor of power allegories. Predicate transformers $\mathcal{P}Y \nrightarrow \mathcal{P}X$ can again be obtained via opposition. The graph functor translates these transformers from **Set** to **Rel**.

While these constructions of modalities depend only on concepts of power allegories and on Peleg composition, the relationships between the different modal operators require closure and duality properties of the inner structure of multirelations and notions of inner determinism and inner functionality (or univalence), which have been studied in the first two parts of this trilogy [FGS23b, FGS23a]. Here we harvest the concepts and results sown in the previous parts to obtain the results outlined above. Our main contributions of this article are part of a conceptual development rather than one or several main theorems. As a first application, this development allows us to prove soundness of a variant of Goldblatt's axiomatisation of concurrent dynamic logic and explain from a structural point of view why and how one of Goldblatt's original axioms needs to be repaired. Using this repaired axiomatisation we present our second application: a soundness proof for a Hoare logic for multirelational programs, where validity of partial-correctness specifications ("Hoare triples") is encoded using Nerode and Wijesekera's box.

Set-theoretic reasoning with multirelations is often tedious; manipulating Peleg's modal operators sometimes requires the axiom of choice. Algebraic reasoning has the potential to tame this complexity and support proof automation with machines. As in [FGS23b, FGS23a], we have used the Isabelle/HOL proof assistant to formalise and check many results in this article, see [GS23], yet without aiming at a complete formalisation. Our article is therefore self-contained without the Isabelle components.

2 Relations and Multirelations

We start with recalling briefly the basics of binary relations and multirelations, building on the first two articles this trilogy [FGS23b, FGS23a]. Our multirelational language is closely related to allegorical and relation-algebraic approaches [SS89, Sch11, FŠ90, BdM97, FKST17]. It blends relation-algebraic concepts with those of boolean power allegories and adds specific concepts for multirelations that have been studied in the first two articles in this trilogy [FGS23b, FGS23a]. We recommend consulting [FGS23b] for definitions and explanations of the basic multirelational concepts used in this article, and [FGS23a] for background on power allegories. Standard relation-algebraic concepts are introduced in the first two articles in the triology as well, but see [SS89, Sch11] for further details. We present more precise links to [FGS23b, FGS23a] across this section.

As mentioned in the introduction, we work in principle in enrichments of the category **Rel** with sets as objects and concrete relations as arrows. We complement set-theoretic definitions by algebraic ones whenever possible and show algebraic proofs whenever we can. We also list all relational and multirelational concepts with respect to a small basis in Appendix A, which extends similar lists in the two predecessor articles. But neither do we have a coherent set of axioms for these concepts in the style of allegories or relation algebras, nor have we attempted to classify our theorems with respect to the concepts and laws they require.

2.1 Binary relations

We write $X \nrightarrow Y$ for the homset $\mathbf{Rel}(X,Y)$, Id_X for the identity relation on X, $\emptyset_{X,Y}$ for the least and $U_{X,Y}$ for the greatest element in $X \nrightarrow Y$, $-R$ for the complement of R and $S - R$ for the relative complement $S \cap -R$, RS for the relational composition of relations R, S of suitable types, R/S and $R\backslash S$ for the left and right residuals of R and S, and $R^\smile$ for the converse of R. We frequently need the modular law $RS \cap T \subseteq (R \cap TS^\smile)S$ and the properties $T\backslash S = (S^\smile/T^\smile)^\smile$, $T/S = -(-TS^\smile)$ and $T\backslash S = -(T^\smile(-S))$ of residuals.

Set-theoretically, for $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$,

$$RS = \{(a,b) \mid \exists c.\ R_{a,c} \wedge S_{c,b}\}, \qquad Id_X = \{(a,a) \mid a \in X\}, \qquad R^\smile = \{(b,a) \mid R_{a,b}\}.$$

Further, for $T : X \nrightarrow Z$ and $S : Y \nrightarrow Z$,

$$T/S = \bigcup\{R : X \nrightarrow Y \mid RS \subseteq T\}.$$

We also need the following concepts: the *symmetric quotient* $T \div S : X \nrightarrow Y$, defined as $T \div S = (T\backslash S) \cap (T^\smile/S^\smile)$, *tests*, which are partial identity relations $R \subseteq Id$, and whose relational composition is intersection, and the *domain* map $dom : (X \nrightarrow Y) \to (X \nrightarrow X)$ defined by

$$dom(R) = Id_X \cap RR^\smile = Id_X \cap RU_{Y,X} = \{(a,a) \mid \exists b.\ R_{a,b}\}.$$

Domain elements and tests form the same full subalgebra of $\mathbf{Rel}(X,X)$ for any X, a complete atomic boolean algebra. The boolean complement of a test P is $\neg P = Id - P$.

Deterministic relations play an important role in our work. The relation $R : X \nrightarrow Y$ is

- *total* if $dom(R) = Id_X$, or equivalently $Id_X \subseteq RR^\smile$,
- *univalent*, or a *partial function*, if $R^\smile R \subseteq Id_Y$,
- *deterministic*, or a *function*, if it is total and univalent.

Functions as deterministic relations in **Rel** are of course graphs of functions in **Set**. We use the relational law $PQ \cap S = (P \cap SQ^\smile)Q$ for univalent Q [SS89] in calculations.

Our definitions of modal operators and proofs for them are based on concepts from power allegories, including monadic concepts in relational form [FŠ90, BdM97]. We summarise them in the following; see [FGS23a, Section 2.1] for details. The isomorphism between relations in $X \nrightarrow Y$ and nondeterministic functions in $X \to \mathcal{P}Y$ in **Set** can be expressed in **Rel** by taking graphs.

The *power transpose*

$$\Lambda(R) = R^\smile \div \in_Y = \{(a, R(a)) \mid a \in X\}$$

maps relations $X \nrightarrow Y$ to functions in $X \nrightarrow \mathcal{P}Y$, where $\in_Y : Y \nrightarrow \mathcal{P}Y$ is the membership relation on Y, which relates each element of Y with the subsets of Y that contain it.

In the other direction, relational postcomposition with the converse of $\in_Y$, the *has-element relation* $\ni_Y : \mathcal{P}Y \nrightarrow Y$, maps relations in $X \nrightarrow \mathcal{P}Y$ to relations in $X \nrightarrow Y$. We henceforth write $\alpha = (-)\ni$. This function plays an important role in this article. It satisfies

$$\alpha(R) = \{(a,b) \mid b \in \bigcup R(a)\}.$$

We also need the *relational image functor* $\mathcal{P} : (X \nrightarrow Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ defined by

$$\mathcal{P}(R) = \Lambda(\ni_X R),$$

which satisfies $\mathcal{P}(R) = \{(A, R(A)) \mid A \subseteq X\}$. It codes the relational image, given by the covariant powerset functor in **Set**, as a graph. It is deterministic by definition.

The unit and multiplication of the monad of the powerset functor in **Set** are recovered in **Rel** as $\eta_X : X \nrightarrow \mathcal{P}X$ and $\mu_X : \mathcal{P}^2X \nrightarrow \mathcal{P}X$ such that $\eta_X = \Lambda(Id_X) = \{(a, \{a\}) \mid a \in X\}$ and $\mu_X = \mathcal{P}(\ni_X)$.

Apart from tests we need the *power test* $P_* : \mathcal{P}X \nrightarrow \mathcal{P}X$ of any test $P \subseteq Id_X$, defined as

$$P_* = (\in_X \backslash P \in_X) \cap Id_{\mathcal{P}X}$$

in [FKST17]. Equivalently, it can be expressed as

$$P_* = (\in_X \backslash PU_{X,\mathcal{P}X}) \cap Id_{\mathcal{P}X} = \{(A, A) \mid \forall a \in A.\ (a, a) \in P\}.$$

Intuitively, if we view the test P as a set, then $P_* = \{(A, A) \mid A \subseteq P\}$. Obviously, $P_* \subseteq Id_{\mathcal{P}X} = (Id_X)_*$.

Finally, for manipulating multirelations, we need

- the *subset relation* $\Omega_Y = \in_Y \backslash \in_Y = \{(A, B) \mid A \subseteq B \subseteq Y\}$ and
- the *complementation relation* $C = \in_Y \div -\in_Y = \{(A, -A) \mid A \subseteq Y\}$.

2.2 Multirelations

A *multirelation* is an arrow $X \nrightarrow \mathcal{P}Y$ in **Rel** and therefore a doubly-nondeterministic function $X \to \mathcal{P}^2Y$ in **Set**. Multirelations allow two levels of nondeterminism and hence two levels of choice: an outer or angelic level given by elements (a, B) and (a, C) of a multirelation, and an inner or demonic level given by the elements $b \in B$ for any (a, B). See [FGS23b] and the references therein for examples and applications.

The *Peleg composition* [Pel87] $* : (X \nrightarrow \mathcal{P}Y) \times (Y \nrightarrow \mathcal{P}Z) \to (X \nrightarrow \mathcal{P}Z)$ can be defined in two steps from the *Kleisli lifting* $(-)_{\mathcal{P}} : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ and the *Peleg lifting* $(-)_* : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ of multirelations [FKST17]:

$$R_{\mathcal{P}} = \mathcal{P}(\alpha(R)), \qquad R_* = dom(R)_* \bigcup_{S \subseteq_d R} S_{\mathcal{P}}, \qquad R * S = RS_*,$$

where, for $R, S : X \nrightarrow Y$, $S \subseteq_d R$ if S is univalent, $dom(S) = dom(R)$ and $S \subseteq R$.

Expanding definitions,

$$\begin{aligned} R_{\mathcal{P}} &= \left\{(A, B) \mid B = \bigcup R(A)\right\}, \\ R_* &= \left\{(A, B) \mid \exists f : X \to \mathcal{P}Y.\ f|_A \subseteq R \wedge B = \bigcup f(A)\right\}, \\ R * S &= \left\{(a, C) \mid \exists B.\ R_{a,B} \wedge \exists f : Y \to \mathcal{P}Z.\ f|_B \subseteq S \wedge C = \bigcup f(B)\right\}. \end{aligned}$$

The Kleisli lifting is the multirelational analogue of the Kleisli lifting or Kleisli extension in the Kleisli category of the powerset monad; see [FGS23a, Section 2.2] for details. It can also be seen as the relational image of the relational approximation of a given multirelation using the map α. By definition, Kleisli liftings of multirelations are functions in **Rel**. Peleg and Kleisli liftings coincide on deterministic multirelations: $R_* = R_{\mathcal{P}}$ if R is deterministic [FKST17].

Remark 2.1. If $R : X \nrightarrow \mathcal{P}Y$ relates an element $a \in X$ with a set $B \subseteq \mathcal{P}Y$, and if $S : Y \nrightarrow \mathcal{P}Z$ relates each $b \in B$ with some $C_b \in Z$, then the Peleg composition $R * S$ relates a with $\bigcup_{b \in B} C_b$. The choice function $f : Y \to \mathcal{P}Z$ in the definition above corresponds to the univalent multirelation

$Y \nrightarrow \mathcal{P}Z$ included in R in the algebraic definition of Peleg lifting and it captures the dependency of the set C_b in the codomain of S on the $b \in B$ from the codomain of R. As multirelations provide a model for alternating nondeterminism, it is no surprise that their Peleg composition corresponds to the composition of transition relations of alternating automata [FS15].

The Peleg lifting thus computes the set of all $(B, \bigcup_{b \in B} C_b)$ such that for all $b \in B$, (b, C_b) is in S. Then $(a, \bigcup_{b \in B} C_b) \in R * S$ if and only if there exists a $B \subseteq Y$ such that $(a, B) \in R$ and $(B, \bigcup_{b \in B} C_b) \in S_*$, which is the case if and only if $(a, \bigcup_{b \in B} C_b) \in RS_*$.

The Peleg composition plays an important role in this article because the modal operators in Peleg's concurrent dynamic logic [Pel87] are based on it. In this context, it models the sequential composition of multirelational programs. Its units are the multirelations η_X; because of this, we henceforth also write 1_X for them.

Peleg composition is not associative – only $(R * S) * T \subseteq R * (S * T)$ holds [FS15] – so that multirelations do not form a category under Peleg composition. Yet it becomes associative if the third factor is univalent [FKST17]; see also [FGS23a] and Section 3 for other situations where multirelations under Peleg composition form categories.

Example 2.2 ([FS15])**.** Associativity fails for the multirelations $R = \{(a, \{a, b\}), (a, \{a\}), (b, \{a\})\}$ and $S = \{(a, \{a\}), (a, \{b\})\}$ on the set $X = \{a, b\}$:

$$\begin{aligned}(R * R) * S &= \{(a, \{a\}), (a, \{b\}), (b, \{a\}), (b, \{b\})\} \\ &\subset \{(a, \{a, b\}), (a, \{a\}), (a, \{b\}), (b, \{a\}), (b, \{b\})\} \\ &= R * (R * S).\end{aligned}$$

Peleg composition preserves arbitrary unions in its first argument and therefore has a right adjoint [FGS23b]

$$R * S = RS_* \subseteq T \Leftrightarrow R \subseteq T/S_* = T/S.$$

where residual notation has been overloaded. It is important for defining multirelational modal box operators in Section 5.

Multirelational modal operators also require multirelational tests. These are subsets of 1_X for any X, and represent assertions in Peleg's concurrent dynamic logic, formalised as multirelational programs that observe a property of program states, but do not alter the state. Like relational tests, they correspond to subsets of X. They are related to relational tests via the isomorphism $(-)1_X$ from relational tests to multirelational ones and its inverse $(-)1_X^{\smile}$, which specialise the isomorphisms Λ and α between relations in **Rel** and nondeterministic functions in **Set** (see Section 3). The boolean complement of a multirelational test P is $\neg P = 1 - P$, overloading notation for the complement of relational tests.

The Peleg lifting of a multirelational test $P \subseteq 1_X$ is $P_* = \{(A, A) \mid \forall a \in A.\ (a, \{a\}) \in P\}$, the power test of the isomorphic relational test below Id_X. This justifies overloading the power test and the Peleg lifting notation.

Lemma 2.3 ([FKST17])**.** *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Id_X$. Then*

$$dom(R_*) = dom(R)_*, \qquad (PR)_* = P_* R_*, \qquad 1_X P_* = P 1_X.$$

Remark 2.4. To simplify notation, we often identify relational and multirelational tests with sets. A power test for the relation or multirelation P is then just the powerset of the set P: $P_* = \mathcal{P}P$. In particular we may assume that $dom(R) = \{a \mid \exists b.\ R_{a,b}\}$ for all $R : X \nrightarrow \mathcal{P}Y$. We write, for instance, $A \subseteq P$ instead of $Id_A \subseteq P$ and $A \cap P = \emptyset$ instead of $Id_A \cap P = \emptyset$. We also write $\neg$ for set complement.

We finish this section with a brief discussion of concepts related to the inner structure of multirelations, which have been studied in detail in [FGS23b, Section 3], based on previous work in [Rew03, FS15, FS16]. We henceforth speak of inner and outer concepts for a clear distinction.

For $R, S : X \nrightarrow \mathcal{P}Y$, we define

- the *inner union* $R \Cup S = \{(a, A \cup B) \mid R_{a,A} \wedge S_{a,B}\}$ with unit $1_{\Cup} = \{(a, \emptyset) \mid a \in X\}$,
- the *inner complementation* $\sim R = RC = \{(a, -A) \mid R_{a,A}\}$,
- the set $\mathsf{A}_{\Cup} = \{(a, \{b\}) \mid a \in X \wedge b \in Y\}$ of *atoms* in $X \nrightarrow \mathcal{P}Y$,
- the *dual operation* $R^{\mathsf{d}} = -{\sim}R = -RC$.

The inner union is associative and commutative, but not generally idempotent. In Peleg's concurrent dynamic logic, it is related to the parallel composition of multirelational programs. These concepts feature in laws for multirelational modal operators in Sections 5 and 6.

Inner deterministic multirelations model an inner or demonic choice of multirelations. The multirelation $R : X \nrightarrow Y$ is

- *inner total* if $R \subseteq -1_{\Cup}$, that is, B is non-empty for each $(a, B) \in R$,
- *inner univalent* if $R \subseteq \mathsf{A}_{\Cup} \cup 1_{\Cup}$, that is, B is either a singleton or empty for each $(a, B) \in R$,
- *inner deterministic* if it is inner total and inner univalent, in which case $B \subseteq Y$ is a singleton set whenever $R_{a,B}$ for some $a \in X$.

We write $\nu(R)$ for the inner total part of R: those pairs in R whose second component is not $\emptyset$, that is, $\nu(R) = R - 1_{\Cup}$. Basic properties of inner total, univalent and deterministic multirelations are studied again in [FGS23b, Section 3]. The structure of inner and outer univalent and deterministic multirelations is the subject of [FGS23a].

Finally, we need the following closures on the inner structure. For $R : X \nrightarrow \mathcal{P}Y$,

- the *up-closure* $R\!\uparrow = R\Omega = \{(a, A) \mid \exists (a, B) \in R.\ B \subseteq A\}$,
- the *down-closure* $R\!\downarrow = R\Omega^{\smile} = \{(a, A) \mid \exists (a, B) \in R.\ A \subseteq B\}$.

See [FGS23b, Section 4] for details. The up-closure and down-closure are related by inner duality with respect to $\sim$.

3 Deterministic Multirelations

In this section, we summarise advanced properties of deterministic multirelations from [FGS23a, FGS23b] that are important for reasoning with modal operators.

Recall that a *quantaloid* is a category in which every homset forms a complete lattice and where arrow composition preserves arbitrary sups in both arguments.

The main result in [FGS23a] on inner and outer deterministic multirelations is as follows.

Proposition 3.1 ([FGS23a, Proposition 3.8]). *The inner deterministic multirelations with $*$, 1, $\bigcup$ and the outer deterministic multirelations with $*$, 1 and $\bigCup$ form quantaloids isomorphic to the quantaloid of binary relations with relational composition, the identity relation and set union.*

For here, it is particularly important to note that Λ is the isomorphism from the quantaloid **Rel** to that of outer deterministic multirelations, which sends relations $X \nrightarrow Y$ to the isomorphic functions $X \to \mathcal{P}Y$ (modelled as outer deterministic multirelations $X \nrightarrow \mathcal{P}Y$ via their graphs), relational composition to Peleg composition, the Id_X to the 1_X and unions to inner unions, which are idempotent on outer deterministic multirelations [FGS23b, Lemma 3.6]. Its inverse is α, which decomposes functions in $X \to \mathcal{P}Y$ (as outer deterministic multirelations $X \nrightarrow \mathcal{P}Y$) into relations $X \nrightarrow Y$.

Likewise, $\eta = (-)1$ is the isomorphism from **Rel** to the quantaloid of inner deterministic multirelations and α is its inverse. The functor η simply wraps all elements in the codomain of a relation into set braces, while α removes these braces. The following consequence of this fact is important for concurrent dynamic logic, in particular Goldblatt's axiomatisation in Section 6.

Lemma 3.2 ([FGS23a, Lemmas 3.4 and 3.15]). *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : Y \nrightarrow \mathcal{P}Z$. Then $\alpha(R * S) \subseteq \alpha(R)\alpha(S)$, and equality holds if R and S are inner or outer deterministic.*

The following counterexample shows that equality does not hold in general.

Example 3.3 ([FGS23a, Example 3.16]). For $R = \{(a, \{a, b\})\}$,

$$\alpha(R * R) = \alpha(\emptyset) = \emptyset \subset \{(a, a), (a, b)\} = \alpha(R) = \alpha(R)\alpha(R).$$

In [FGS23a, Section 3.3], we have also introduced determinisation maps for multirelations. While α approximates multirelations $X \nrightarrow \mathcal{P}Y$ by relations $X \nrightarrow Y$, these maps send them to the isomorphic inner and outer nondeterministic multirelation instead. Let $R : X \nrightarrow \mathcal{P}Y$ be a multirelation. The *outer determinisation* operation $\delta_o = \Lambda \circ \alpha$ sends R to the outer deterministic multirelation isomorphic to the relation $\alpha(R)$. The *inner determinisation* operation $\delta_i = \eta \circ \alpha$ sends R to the inner deterministic multirelation isomorphic to $\alpha(R)$. They satisfy

$$\delta_o(R) = \{(a, B) \mid B = \bigcup R(a)\} \qquad \text{and} \qquad \delta_i(R) = \{(a, \{b\}) \mid b \in \bigcup R(a)\}.$$

Proposition 3.4 ([FGS23a, Corollary 3.10]). *The maps δ_i and δ_o are isomorphisms between the quantaloids of inner deterministic and outer deterministic multirelations.*

By functoriality, $\delta_i(R*S) = \delta_i(R)*\delta_i(S)$ if R, S are outer deterministic, and $\delta_o(R*S) = \delta_o(R)* \delta_o(S)$ if R, S are inner deterministic. Moreover, the inner and outer deterministic multirelations are precisely the fixpoints of δ_i and δ_o, respectively [FGS23a, Corollary 3.11].

We end this section with a collection of standard properties of power allegories from [BdM97] that are helpful in proofs in the remaining sections. They are also listed in [FGS23a, Lemmas 2.1 and 2.2].

Lemma 3.5. *Let $R : X \nrightarrow Y$, $S : Y \nrightarrow Z$ and let $f, g : X \nrightarrow Y$ be outer deterministic. Then*

1. $\alpha(\Lambda(R)) = R$ *and* $\Lambda(\alpha(f)) = f$,
2. $g\Lambda(S) = \Lambda(gS)$ *and* $\Lambda(\ni_X) = Id_{\mathcal{P}X}$,
3. $\Lambda(RS) = \Lambda(R)\mathcal{P}(S)$,
4. $\Lambda(f) = f\eta$,
5. $\alpha(\eta) = Id$.

4 Properties of Tests and Domain

The study of modal operators in a multirelational setting is based on properties of tests and domain, which we consider in this section. Recall from Section 2 that we usually identify relational and multirelational tests with sets instead of considering them as subsets of Id or 1 explicitly. Further, Peleg compositions $P * R$ and $R * P$ of multirelational tests P and multirelations R can always be replaced by relational compositions $\alpha(P)R$ and RP_*, respectively, where $\alpha(P) = P1^{\smile}$ is the isomorphic relational test and P_* the isomorphic power test of P.

Lemma 4.1. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then*

1. $RP_* = R \cap UP_*$ *and* $R\neg(P_*) = R - UP_*$,
2. $(UP_*)\downarrow = UP_*$,
3. $\nu(RP_*) = \nu(R)P_*$,
4. $\alpha(R)\eta(P) = \delta_i(R)P_* = \delta_i(\nu(R\downarrow)P_*)$.

Proof. Item (1) is standard in relation algebra.

For (2), it suffices to show that $P_*\Omega^\smile \subseteq UP_*$. This follows from $P_*\Omega^\smile\ni \subseteq UP$ because $UP_* = U(\in\backslash PU \cap Id) = (\in\backslash PU)^\smile = UP/\ni$, and in fact from $P_*\ni \subseteq UP$ as $\Omega^\smile\ni = \ni$. Indeed, $\in P_* \subseteq \in(\in\backslash PU) = PU$ and the claim follows from properties of converse.

For (3), $\nu(RP_*) = (R \cap UP_*) - 1_\Cup = (R - 1_\Cup) \cap UP_* = \nu(R)P_*$ using (1).

For (4), $\alpha(R)P1 = \alpha(R)1P_* = \delta_i(R)P_*$ yields the first equation, using Lemma 2.3 and expanding definitions. For the second equation, $\delta_i(R) = R{\downarrow} \cap \mathsf{A}_\Cup$ by [FGS23a, Lemma 3.14]. It follows that $\delta_i(R)P_* = (R{\downarrow} \cap UP_*){\downarrow} \cap \mathsf{A}_\Cup = \delta_i(R{\downarrow}P_*) = \delta_i(\nu(R{\downarrow}P_*)) = \delta_i(\nu(R{\downarrow})P_*)$. The first step uses (1), (2) and the fact that intersections of down-closed sets are down-closed, the second (1) and the above characterisation of δ_i, the third $\delta_i \circ \nu = \delta_i$ ([FGS23a, Lemma 4.1(2)]) and the last (3). □

Tests $P \subseteq Id_X$ obviously form a boolean algebra with complementation $\neg P = Id_X - P$. Tests $P_* \subseteq Id_{\mathcal{P}X}$ thus form a boolean algebra with $\neg(P_*) = Id_{\mathcal{P}X} - P_*$.

Domain operations on algebras of multirelations have been studied in [FS15, FS16]. The standard definition for relations in $X \nrightarrow Y$ in Section 2 applies to multirelations.

Recall that $dom(R_*) = dom(R)_*$ by Lemma 2.3; also note that $dom(P) = P = dom(\eta(P))$ for tests P. Another important property of domain is *domain locality* [FS15],

$$dom(RS_*) = dom(R\, dom(S)_*).$$

Further, $dom(R \cap S) = 1 \cap SR^\smile$ [FŠ90], $dom(PR_*) = P \cap dom(R)$, and, set-theoretically,

$$dom(RP_*) = \{a \mid \exists B.\ R_{a,B} \wedge B \subseteq P\} \qquad \text{and} \qquad dom(R_*) = \{A \mid \forall a \in A.\ \exists B.\ R_{a,B}\}.$$

Lemma 4.2. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then*

1. $dom(\delta_i(R)) = dom(\alpha(R)) = dom(\nu(R)) = dom(\nu(\delta_o(R)))$,
2. $dom(\alpha(R)P) = dom(\nu(R{\downarrow})P_*)$ *and* $dom(\alpha(R)\neg P) = dom(R\neg(P_*))$.

Proof. For (1), first $dom(\delta_i(R)) = dom(\alpha(R)1) = dom(\alpha(R)dom(1)) = dom(\alpha(R))$. Second, $dom(\alpha(R)) = Id \cap R\ni U = Id \cap R(-1_\Cup)^\smile = dom(R - 1_\Cup) = dom(\nu(R))$ because $U\in = -1_\Cup$ holds by a simple set-theoretic calculation. Third, $dom(\nu(\delta_o(R))) = dom(\delta_i(\delta_o(R))) = dom(\delta_i(R))$ using the previous identities and $\delta_i \circ \delta_o = \delta_i$ [FGS23a, Lemma 3.17].

For (2), the first equality follows from (1) and Lemma 4.1(3) and (4) using that $\nu\circ\nu = \nu$ [FS16]. For the second, $\neg dom(\ni\neg P) = Id - (\ni\neg PU) = Id - (\ni - PU) = Id \cap (\in\backslash PU) = P_*$, and therefore $dom(\alpha(R)\neg P) = dom(R\, dom(\ni\neg P)) = dom(R\neg(P_*))$. □

From domain properties we immediately obtain $dom(\delta_i(R)) = dom(R\, dom(\ni 1)) \subseteq dom(R)$ and $dom(\alpha(R)) = dom(R\, dom(\ni)) \subseteq dom(R)$. Intuitively, $dom(\delta_i(R)) = dom(\alpha(R))$ follows immediately from the isomorphism between relations and inner deterministic multirelations. Yet one needs to remove the elements that are mapped to $\emptyset$ by (the function isomorphic to) $\delta_o(R)$ using ν to make the resulting domain equal to that of $\alpha(R)$ and $\delta_i(R)$. This explains the identities in Lemma 4.2(1).

The identities in Lemma 4.2(2) are less intuitive, but needed for deriving properties of multirelational modal operators below.

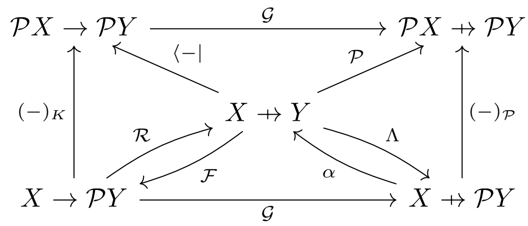
5 Modal Operators

We now turn to modal box and diamond operators for multirelations, but first recall the standard relational modalities, generalising the underlying Kripke frames of type $X \nrightarrow X$ to $X \nrightarrow Y$ and the operators from $\mathcal{P}X \to \mathcal{P}X$ to $\mathcal{P}X \to \mathcal{P}Y$ or $\mathcal{P}Y \to \mathcal{P}X$.

5.1 Relational modal operators

The backward relational diamond operator $\langle -| : (X \nrightarrow Y) \to (\mathcal{P}X \to \mathcal{P}Y)$ can be obtained directly from the relational image operation: $\langle R|P = cod(PR) = dom((PR)^{\smile})$ for all $R : X \nrightarrow Y$ and $P \subseteq X$. Its opposite forward relational diamond operator $|-\rangle : (X \nrightarrow Y) \to (\mathcal{P}Y \to \mathcal{P}X)$ is given as $|R\rangle = \langle R^{\smile}|$ so that $|R\rangle Q = dom(RQ)$ for all $R : X \nrightarrow Y$ and $Q \subseteq Y$. Forward and backward box operators $|-] : (X \nrightarrow Y) \to (\mathcal{P}Y \to \mathcal{P}X)$ and $[-| : (X \nrightarrow Y) \to (\mathcal{P}Y \to \mathcal{P}X)$ can be obtained from $|-\rangle$ and $\langle -|$ by De Morgan duality: $|R] = \neg \circ |R\rangle \circ \neg$ and $[R| = \neg \circ \langle R| \circ \neg$. They are again related by opposition $|R] = [R^{\smile}|$.

Using nondeterministic functions – arrows of the Kleisli category $\mathbf{Set}_{\mathcal{P}}$ of the powerset monad in $\mathbf{Set}$ – instead, $\langle -| = (-)_K$, the Kleisli extension of such arrows. The left triangle in the diagram below thus commutes, where $\mathcal{F} : \mathbf{Rel} \to \mathbf{Set}_{\mathcal{P}}$ and $\mathcal{R} : \mathbf{Set}_{\mathcal{P}} \to \mathbf{Rel}$ indicate the isomorphism between the two categories.



The graph functor $\mathcal{G} : \mathbf{Set} \to \mathbf{Rel}$ translates this diagram to $\mathbf{Rel}$. The relational image operator $\mathcal{P}$ and the Kleisli lifting $(-)_{\mathcal{P}}$ in $\mathbf{Rel}$ now play the role of $\langle -|$ and $(-)_K$ in $\mathbf{Set}$. The Kleisli extension $(-)_K$ is actually an isomorphism between $\mathbf{Set}_{\mathcal{P}}$ and the category of backward diamond operators (with inverse given by postcomposition with the unit of the powerset monad). It preserves the entire quantaloid structure in $\mathbf{Set}_{\mathcal{P}}$ that comes from $\mathbf{Rel}$. The standard definition of the Kleisli lifting translates to multirelations as $(-)_{\mathcal{P}} = \mathcal{P}(-)\mu$ [FGS23a, Lemma 2.4]. Therefore, and of course owing to the composition of isomorphisms along the other outer faces of our diagram, we obtain a corresponding isomorphism between categories of outer deterministic multirelations and their Kleisli liftings on the right of this diagram. All arrows in the diagram are therefore isomorphisms between categories.

In the following, we focus on the forward operators $|-\rangle$ and $|-]$ and use the more conventional diamond notation $\langle -\rangle_r$ and $[-]_r$ with indices indicating their relational nature. Hence, for $R : X \nrightarrow Y$ and $Q \subseteq Y$,

$$\langle R\rangle_r Q = dom(RQ) = \{a \mid \exists b \in Y.\ R_{a,b} \wedge b \in Q\},$$
$$[R]_r Q = \neg dom(R\neg Q) = \{a \mid \forall b \in Y.\ R_{a,b} \Rightarrow b \in Q\}.$$

For backward box and diamond operators we simply write $\langle R^{\smile}\rangle_r$ and $[R^{\smile}]_r$. It follows that $\langle R^{\smile}\rangle_r P = \{b \mid \exists a \in X.\ R_{a,b} \wedge a \in P\}$ and $[R^{\smile}]_r P = \{b \mid \forall a \in X.\ R_{a,b} \Rightarrow a \in P\}$. This describes once again the left of the above diagram.

On the right of the diagram above it is routine to check that the graphs of $\langle R\rangle_r Q$ and $[R]_r P$ are given by

$$\mathcal{P}(R^{\smile}) = \{(Q, P) \mid P = \langle R\rangle_r Q\} \qquad \text{and} \qquad \Lambda(\ni/R) = \{(Q, P) \mid P = [R]_r Q\}.$$

We henceforth write

$$\langle R\rangle_r^{\mathcal{G}} = \mathcal{P}(R^{\smile}) \qquad \text{and} \qquad [R]_r^{\mathcal{G}} = \Lambda(\ni/R),$$

where the superscript $^{\mathcal{G}}$ indicates that these operators are graphs of $\langle R\rangle_r$ and $[R]_r$. The fact that $[R]_r^{\mathcal{G}}$ is an allegorical box operator in $\mathbf{Rel}$ is known [BdM97].

5.2 Multirelational modal operators

Two extensions of relational modalities to multirelations have been proposed. The first replaces relational composition by Peleg composition in relational image operations and their De Morgan duals. It has been advocated by Peleg in his concurrent dynamic logic [Pel87] and studied further in [FS15, FS16]. The second has been proposed implicitly by Nerode and Wijesekera [NW90] and studied further by Goldblatt [Gol92]. Here we show that, in the language of power allegories, it uses α to approximate multirelations by relations in relational boxes and diamonds. In the remainder of this section we formalise and relate these two approaches. We also study the second approach in **Rel**.

First we recall the multirelational modal operators $\langle - \rangle_*, [-]_* : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}Y \to \mathcal{P}X)$ used by Peleg in concurrent dynamic logic. In our semantic setting, they can be defined, for all $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$, as

$$\langle R \rangle_* P = dom(R * P) \qquad \text{and} \qquad [R]_* P = \neg dom(R * \neg P).$$

As already mentioned, these are essentially relational modalities with relational composition replaced by Peleg composition. Indeed, $\langle R \rangle_* P = \langle R \rangle_r P_*$ and $[R]_* P = [R]_r \neg((\neg P)_*)$, where the double negation in the last identity cannot be eliminated. Further, $\langle R \rangle_* P = \neg [R]_* \neg P$ and $[R]_* P = \neg \langle R \rangle_* \neg P$, and, unfolding definitions,

$$\langle R \rangle_* P = \{a \mid \exists B.\ R_{a,B} \wedge B \subseteq P\} \qquad \text{and} \qquad [R]_* P = \{a \mid \forall B.\ R_{a,B} \Rightarrow B \cap P \neq \emptyset\}.$$

As pointed out in the introduction, Nerode and Wijesekera have argued that the semantics of $[-]_*$, as expanded above, is quite different from that of its relational counterpart, and that it fails to capture standard program correctness specifications that are usually associated with modal boxes. They have therefore proposed an alternative multirelational box, to which we add a De Morgan dual diamond operator $\langle - \rangle_\alpha : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}Y \to \mathcal{P}X)$. In the language of power allegories and for $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$, the two operators are

$$[R]_\alpha = [\alpha(R)]_r \qquad \text{and} \qquad \langle R \rangle_\alpha = \langle \alpha(R) \rangle_r.$$

Unfolding definitions yields

$$[R]_\alpha P = \{a \mid \forall B.\ R_{a,B} \Rightarrow B \subseteq P\} \qquad \text{and} \qquad \langle R \rangle_\alpha P = \{a \mid \exists B.\ R_{a,B} \wedge B \cap P \neq \emptyset\},$$

which confirms in particular that $[R]_\alpha P$ is consistent with Nerode and Wijesekera's definition.

These operations are also related by De Morgan duality: $\langle R \rangle_\alpha P = \langle \alpha(R) \rangle_r P = \neg [\alpha(R)]_r \neg P = \neg [R]_\alpha \neg P$ and likewise $[R]_\alpha P = \neg \langle R \rangle_\alpha \neg P$.

Further, $[R]_\alpha P = \neg \langle R \rangle_r \neg (P_*) = [R]_r P_*$ by Lemma 4.2(2), and then $\langle R \rangle_\alpha P = \langle R \rangle_r \neg((\neg P)_*)$ by De Morgan duality. Once again, the double negation in the last identity cannot be eliminated.

5.3 Properties of multirelational modal operators

In this section we present alternative algebraic definitions of Nerode and Wijesekera's box operator; the richness of the multirelational language offers many possibilities. We also present some simple relationships between the different kinds of modalities.

Lemma 4.1(1) leads directly to a different, purely multirelational definition of $[-]_\alpha$.

Lemma 5.1. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then $[R]_\alpha P = \neg dom(R - UP_*)$.*

Further, $[-]_\alpha$ and $\langle - \rangle_\alpha$ can be defined in various ways in terms of other modalities.

Lemma 5.2. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then*

1. $[R]_\alpha = [\nu(R{\downarrow})]_* = [\delta_i(R)]_* = [\delta_o(R)]_\alpha = [\delta_i(R)]_\alpha = \langle \delta_o(R) \rangle_*$,
2. $\langle R \rangle_\alpha = \langle \nu(R{\downarrow}) \rangle_* = \langle \delta_i(R) \rangle_* = \langle \delta_o(R) \rangle_\alpha = \langle \delta_i(R) \rangle_\alpha = [\delta_o(R)]_*$.

Proof. For (1), first $[R]_\alpha P = [\alpha(R)]_r P = \neg dom(\nu(R{\downarrow})(\neg P)_*) = [\nu(R{\downarrow})]_* P$ using Lemma 4.2(2). Second, $[R]_\alpha P = \neg dom(\alpha(R) \neg P) = \neg dom(\delta_i(R)(\neg P)_*) = [\delta_i(R)]_r \neg((\neg P)_*) = [\delta_i(R)]_* P$ using Lemma 4.1(4). The proofs of the third and fourth identity are trivial. Finally,

$$[\delta_o(R)]_\alpha P = [\delta_o(R)]_r P_* = \langle \delta_o(R) \rangle_r P_* = \langle \delta_o(R) \rangle_* P$$

because $\delta_o(R)$ is outer deterministic, so the coincidence of relational boxes and diamonds is standard. The proofs for (2) follow from (1) by De Morgan dualities. □

Lemma 5.2 implies the following fact.

Corollary 5.3. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then $\langle \nu(R)\rangle_* P \subseteq \langle R\rangle_\alpha P$ and $[R]_\alpha P \subseteq [\nu(R)]_* P$.*

Its proof is immediate from the standard fact that relational diamonds preserve $\subseteq$ in both arguments, while relational boxes reverse this order in their first argument (and preserve it in their second one), and relational representations of multirelational modal operators.

Lemma 5.2 also indicates situations where multirelational modalities coincide.

Corollary 5.4. *Let $R : X \nrightarrow \mathcal{P}Y$. Then*

1. *$[R]_\alpha = [R]_*$ and $\langle R\rangle_\alpha = \langle R\rangle_*$ if R is inner deterministic,*
2. *$[R]_\alpha = \langle R\rangle_*$ and $[R]_* = \langle R\rangle_\alpha$ if R is outer deterministic.*

The proofs use properties of Lemma 5.2 together with fixpoint properties of inner and outer deterministic multirelations and properties of δ_o and δ_i from [FGS23a, FGS23b].

Remark 5.5. The multirelational boxes and diamonds specialise to relational ones. Let $R : X \nrightarrow Y$. Then $[\eta(R)]_\alpha = [\alpha(\eta(R))]_r = [R]_r$ and $\langle \eta(R)\rangle_\alpha = \langle \alpha(\eta(R))\rangle_r = \langle R\rangle_r$ is a trivial consequence of the fact that η and α form a bijective pair. Further, $\langle \eta(R)\rangle_* P = dom(R1P_*) = dom(RP1) = dom(RP) = \langle R\rangle_r P$ using Lemma 2.3. The identity $[\eta(R)]_* = [R]_r$ then follows by duality.

Here is another definition of $[-]_\alpha$.

Lemma 5.6. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then $[R]_\alpha P = \neg dom(\delta_o(R)\neg(P_*))$.*

Proof. $[R]_\alpha P = [\delta_o(R)]_\alpha P = [\delta_o(R)]_r P_* = \neg dom(\delta_o(R)\neg(P_*))$ using Lemma 5.2. □

Our final definition of $[-]_\alpha$ requires a technical lemma.

Lemma 5.7. *Let $R : X \nrightarrow \mathcal{P}Y$, $P \subseteq X$ and $Q \subseteq Y$. Then $P \subseteq [R]_\alpha Q \Leftrightarrow PR \subseteq RQ_*$.*

Proof. Using P and Q as relational or multirelational tests depending on the context,

$$P \subseteq [R]_\alpha Q \Leftrightarrow P \subseteq \neg dom(R\neg(Q_*)) \Leftrightarrow P dom(R\neg(Q_*)) \subseteq \emptyset \Leftrightarrow PR\neg(Q_*) \subseteq \emptyset \Leftrightarrow PR \subseteq RQ_*.$$

The penultimate step uses a standard property of *dom* of relations. □

Lemma 5.8. *Let $R : X \nrightarrow \mathcal{P}Y$ and $P \subseteq Y$. Then $[R]_\alpha P = (UP_*)/R \cap 1 = (RP_*)/R \cap 1$.*

Proof. For the first identity we use Lemmas 4.1 and 5.7 and the Galois connection for residuation:

$$\begin{aligned} Q \subseteq [R]_\alpha P &\Leftrightarrow QR \subseteq RP_* \\ &\Leftrightarrow QR \subseteq R \cap UP_* \\ &\Leftrightarrow QR \subseteq UP_* \\ &\Leftrightarrow Q \subseteq (UP_*)/R \\ &\Leftrightarrow Q \subseteq (UP_*)/R \cap 1. \end{aligned}$$

Thus $[R]_\alpha P = (UP_*)/R \cap 1$. The proof of the second identity is similar. □

The preceding definitions of $[-]_\alpha$ are useful for establishing further properties. Identities such as $[\delta_i(R)]_\alpha = [\delta_o(R)]_\alpha$ are instances of a more general result. To this end we consider sufficient conditions for a function f to satisfy $[R]_\alpha = [f(R)]_\alpha$ or $[R]_r = [f(R)]_\alpha$:

1. Note that $[R]_\alpha = [\alpha(R)]_r = [\alpha(f(R))]_r = [f(R)]_\alpha$ if $\alpha \circ f = \alpha$. The latter holds, for example, for $f = \downarrow$ or $f = \nu$ because $\alpha(R\downarrow) = \alpha(R)$ [FGS23a, Lemma 3.15(3)] and $\alpha \circ \nu = \alpha$ [FGS23a, Lemma 4.1(1)].

2. If $[R]_\alpha = g(f(R))$ for an arbitrary function g describing the context, then

$$[R]_\alpha = g(f(R)) = g(f(f(R))) = [f(R)]_\alpha$$

if f is idempotent. This situation arises, for example, for $g = [-]_*$ and $f = \delta_i$ or $f = \nu(-\downarrow)$, and for $g = [\nu(-)]_*$ and $f = \downarrow$ by [FGS23a, Lemma 3.17] and Lemma 5.2. Another instance has $f = \delta_o$ according to Lemma 5.6.

3. If $[R]_\alpha = [g(R)]_\alpha$ has already been established (for example, by the above instances) and $g \circ f = g$, then $[R]_\alpha = [g(R)]_\alpha = [g(f(R))]_\alpha = [f(R)]_\alpha$ generalising the first pattern. Combinations of δ_i, δ_o and ν that give rise to instances can be found in [FGS23a, Lemmas 3.17 and 4.1].

4. If $[R]_\alpha = [g(R)]_r$ has already been established and f is a right-inverse of g, then

$$[R]_r = [g(f(R))]_r = [f(R)]_\alpha.$$

For example, Λ is a right-inverse of α, hence $[R]_r = [\Lambda(R)]_\alpha$ for any relation R.

Remark 5.9. The above instances imply that $[R]_\alpha = [\delta_o(R)]_\alpha = [\delta_i(R)]_\alpha = [R\downarrow]_\alpha = [\nu(R)]_\alpha$ and further equalities. All of the arguments R, $\delta_o(R)$, $\delta_i(R)$, $R\downarrow$ and $\nu(R)$ contain essentially the same information (as regards box) just arranged differently. Yet the different arguments have different properties, so switching between them can be useful. For example, $\delta_o(-)\downarrow$ and $\downarrow$ are closure operators with respect to $\subseteq$; ν is an interior operator with respect to $\subseteq$; other options give closure/interior operators or extrema with respect to certain (pre)orders.

5.4 Multirelational modal operators as relations

We now return to the right-hand triangle in the diagram from the beginning of this section and study the relational and multirelational modal operators in **Rel** instead of **Set**, hence entirely in our multirelational language.

First we consider graphs of the relational modal operators $[-]_r$ and $\langle - \rangle_r$. As already mentioned, they are given by $[-]_r^{\mathcal{G}} = \Lambda(\ni/(-))$ and $\langle - \rangle_r^{\mathcal{G}} = \mathcal{P}((-)^\smile)$. More concretely, rewriting the definitions at the beginning of this section shows that, for $R : X \nrightarrow Y$, $P \subseteq X$ and $Q \subseteq Y$,

$$\langle R \rangle_r^{\mathcal{G}} = \{(Q, P) \mid P = \langle R \rangle_r Q\} \qquad \text{and} \qquad [R]_r^{\mathcal{G}} = \{(Q, P) \mid P = [R]_r Q\}.$$

Backward modalities $\langle R^\smile \rangle$ and $[R^\smile]$ correspond in an analogous way to $\mathcal{P}(R)$ and $\Lambda(\ni/R^\smile)$. The extension to the multirelational modalities $\langle - \rangle_\alpha^{\mathcal{G}}$ and $[-]_\alpha^{\mathcal{G}}$ is then straightforward by inserting α's: for $R : X \nrightarrow \mathcal{P}Y$, $P \subseteq X$ and $Q \subseteq Y$, $\langle R \rangle_\alpha^{\mathcal{G}} = \langle \alpha(R) \rangle_r^{\mathcal{G}}$ and $[R]_\alpha^{\mathcal{G}} = [\alpha(R)]_r^{\mathcal{G}}$, and therefore

$$\langle R \rangle_\alpha^{\mathcal{G}} = \{(Q, P) \mid P = \langle R \rangle_\alpha Q\} \qquad \text{and} \qquad [R]_\alpha^{\mathcal{G}} = \{(Q, P) \mid P = [R]_\alpha Q\}.$$

This explains in particular the role of the Kleisli lifting $(-)_\mathcal{P} = \mathcal{P} \circ \alpha$ as a backward diamond operator $(X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ relative to its standard counterpart $(-)_K : (X \to \mathcal{P}Y) \to (\mathcal{P}X \to \mathcal{P}Y)$ in our diagram more formally.

For any relation $X \nrightarrow Y$ or multirelation $X \nrightarrow \mathcal{P}Y$ we can use these correspondences for translating properties from the maps $\mathcal{P}X \to \mathcal{P}Y$ to their graphs $\mathcal{P}X \nrightarrow \mathcal{P}Y$, at least in tabular allegories.

Alternatively we can also reason about the latter in point-free style in the extended language of power allegories used in previous sections. This uses the subset relation Ω and the complement relation C from relation algebra, introduced at the end of Section 2.1, rather strongly. We need in particular standard properties from relation algebra such as $-{\in}C = {\in}$ and ${\in}C = -{\in}$, $C = C^\smile$, $C^2 = Id$ and in particular that $R(-S) = -(RS)$ if and only if relation R is outer deterministic. We also recall a lemma from [FGS23a], to which we add a third property.

Lemma 5.10. *Let $R : X \nrightarrow Y$. Then*

1. $\Lambda(R)C = \Lambda(-R)$,
2. $\Lambda(R)\Omega = R^{\smile}\backslash\in = (\ni/R)^{\smile}$,
3. $\ni/R = (R\in)^{\mathsf{d}\smile}$.

Proof. Items (1) and (2) are proved in [FGS23a, Lemma 2.3]. For (3), $(R\in)^{\mathsf{d}\smile} = -(C(R\in)^{\smile}) = -(-\ni R^{\smile}) = \ni/R$, using the definition of the dual operator from $(-)^{\mathsf{d}}$, introduced in Section 2.2, in the first step. □

Using the dual operator allows us to rewrite the definition of the relational box operator without using residuation.

Corollary 5.11. *Let $R : X \nrightarrow Y$. Then $[R]^{\mathcal{G}}_r = \Lambda((R\in)^{\mathsf{d}\smile})$ and $\langle R\rangle^{\mathcal{G}}_r = \Lambda((R\in)^{\smile})$.*

The second identity is trivial, but it shows the correspondence with the first.

Next we derive the standard relationships between boxes and diamonds to indicate how such properties can be expressed in **Rel**. In (1) below we prove the De Morgan duality between boxes and diamonds. In (2) and (3) we translate the standard conjugations and Galois connections for relational boxes and diamonds to **Rel**:

$$\langle R\rangle_r P \subseteq \neg Q \Leftrightarrow \langle R^{\smile}\rangle_r Q \subseteq \neg P \qquad \text{and} \qquad \langle R^{\smile}\rangle_r P \subseteq Q \Leftrightarrow P \subseteq [R]_r Q.$$

Proposition 5.12. *Let $R : X \nrightarrow Y$. Then*

1. $C\langle R\rangle^{\mathcal{G}}_r C = [R]^{\mathcal{G}}_r$ *and* $\langle R\rangle^{\mathcal{G}}_r = C[R]^{\mathcal{G}}_r C$,
2. $C(\langle R\rangle^{\mathcal{G}}_r \Omega)^{\smile} = \langle R^{\smile}\rangle^{\mathcal{G}}_r C\Omega^{\smile}$,
3. $[R]^{\mathcal{G}}_r \Omega^{\smile} = (\langle R^{\smile}\rangle^{\mathcal{G}}_r \Omega)^{\smile}$.

Proof. For (1), $C\mathcal{P}(R^{\smile})C = \Lambda(-\ni)\mathcal{P}(R^{\smile})C = \Lambda(-\ni R^{\smile})C = \Lambda(-(-\ni R^{\smile})) = \Lambda(\ni/R)$. The first step unfolds C, the second uses Lemma 3.5(3), the third Lemma 5.10(1) and the fourth a standard property of residuals. The second identity in (1) then follows from $C^2 = Id$.

For (2),

$$\begin{aligned}
\mathcal{P}(R)C\Omega^{\smile} &= \Lambda(-(\ni R))(-(-\ni\in)) \\
&= -(\Lambda(-(\ni R))(-\ni)\in) \\
&= -(\Lambda(-(\ni R))C\ni\in) \\
&= -(\Lambda(\ni R)\ni\in) \\
&= -(\ni R\in) \\
&= -(C(-\ni)(\ni R^{\smile})^{\smile}) \\
&= C(\ni/(\ni R^{\smile})) \\
&= C(\mathcal{P}(R^{\smile})\Omega)^{\smile}
\end{aligned}$$

The first step rewrites $\mathcal{P}$ and $\Omega^{\smile}$, and applies Lemma 5.10(1). The second uses outer determinism of $\Lambda(-(\ni R))$. The third step uses standard properties of C and the fourth uses again Lemma 5.10(1). The fifth step uses Lemma 3.5(1) and the sixth uses again standard properties of C. The seventh uses outer determinism of C and standard properties of residuation. The final step uses Lemma 5.10(2) and properties of residuation.

Finally, (3) is immediate from (1) and (2) using $C^2 = Id$. □

As expected, the relation C plays the role of $\neg$ in **Rel** and Ω that of $\subseteq$. The results in Proposition 5.12 translate to $\langle -\rangle^{\mathcal{G}}_\alpha$ and $[-]^{\mathcal{G}}_\alpha$ by instantiation with $\alpha(R)$.

Next we show that the multirelational box $[-]^{\mathcal{G}}_\alpha$ can be expressed using the superset relation $\Omega^{\smile}$ and without residuation.

Lemma 5.13. *Let $R : X \nrightarrow \mathcal{P}Y$. Then $[R]^{\mathcal{G}}_{\alpha} = \Lambda(\Omega^{\smile}/R) = \Lambda((R{\ni}{\in})^{\mathsf{d}\smile})$ and $\langle R\rangle^{\mathcal{G}}_{\alpha} = \Lambda((R{\ni}{\in})^{\smile})$.*

Proof. For the first identity, $\Omega^{\smile}/R = ({\ni}/{\ni})/R = {\ni}/R{\ni} = {\ni}/\alpha(R)$, where the second step uses a standard "currying" property of residuals. The remaining identities are immediate from Corollary 5.11. □

Peleg's modal operators $\langle -\rangle_*$ and $[-]_*$ can of course be translated to **Rel** and expressed as graphs as well. For $R : X \nrightarrow \mathcal{P}Y$ it is routine to check that

$$\Lambda(R{\uparrow}^{\smile}) = \{(Q, P) \mid P = \langle R\rangle_* Q\} \qquad \text{and} \qquad \Lambda(R{\uparrow}^{\mathsf{d}\smile}) = \{(Q, P) \mid P = [R]_* Q\}.$$

Hence $\Lambda(R{\uparrow}^{\smile})$ is the analogue of $\langle R\rangle_*$ induced by the graph functor, and $\Lambda(R{\uparrow}^{\mathsf{d}\smile})$ the analogue of $[R]_*$. We can therefore write

$$\langle R\rangle^{\mathcal{G}}_* = \Lambda(R{\uparrow}^{\smile}) \qquad \text{and} \qquad [R]^{\mathcal{G}}_* = \Lambda(R{\uparrow}^{\mathsf{d}\smile}).$$

Once again we obtain the De Morgan dualities expected, using C instead of $\neg$.

Proposition 5.14. *Let $R : X \nrightarrow \mathcal{P}Y$. Then $\langle R\rangle^{\mathcal{G}}_* = C[R]^{\mathcal{G}}_* C$ and $[R]^{\mathcal{G}}_* = C\langle R\rangle^{\mathcal{G}}_* C$.*

Proof. $C\Lambda(R{\uparrow}^{\smile})C = \Lambda(-(CR{\uparrow}^{\smile})) = \Lambda(R{\uparrow}^{\mathsf{d}\smile})$, using Lemmas 3.5(2) and 5.10(1) in the first step and the definition of $(-)^{\mathsf{d}}$ in the second. The other identity follows using $C^2 = Id$. □

Yet we do not derive Galois connections or conjugations for Peleg's box operators as they do not interact nicely with relational converse.

As a final exercise, we show how the interactions between the different modal operators, which we considered in Lemma 5.2 and Remark 5.5 for **Set**, can be derived in **Rel**. First, $\langle -\rangle^{\mathcal{G}}_{\alpha}$, $\langle -\rangle^{\mathcal{G}}_*$, $[-]^{\mathcal{G}}_{\alpha}$ and $[-]^{\mathcal{G}}_*$ specialise once again to relational $\langle -\rangle^{\mathcal{G}}_r$ and $[-]^{\mathcal{G}}_r$, by analogy to Remark 5.5. This can be proved in our multirelational language, as the following lemma shows. Likewise we show how (some of) the properties from Lemma 5.2 can be derived in this language.

Lemma 5.15. *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : X \nrightarrow Y$. Then*

1. $\langle S\rangle^{\mathcal{G}}_r = \langle \Lambda(S)\rangle^{\mathcal{G}}_{\alpha} = \langle \eta(S)\rangle^{\mathcal{G}}_{\alpha}$ *and* $[S]^{\mathcal{G}}_r = [\Lambda(S)]^{\mathcal{G}}_{\alpha} = [\eta(S)]^{\mathcal{G}}_{\alpha}$,
2. $\langle R\rangle^{\mathcal{G}}_{\alpha} = \langle \delta_o(R)\rangle^{\mathcal{G}}_{\alpha} = \langle \delta_i(R)\rangle^{\mathcal{G}}_{\alpha}$ *and* $[R]^{\mathcal{G}}_{\alpha} = [\delta_o(R)]^{\mathcal{G}}_{\alpha} = [\delta_i(R)]^{\mathcal{G}}_{\alpha}$,
3. $\langle S\rangle^{\mathcal{G}}_r = \langle S{\in}\rangle^{\mathcal{G}}_* = \langle \eta(S)\rangle^{\mathcal{G}}_*$ *and* $[S]^{\mathcal{G}}_r = [S{\in}]^{\mathcal{G}}_* = [\eta(S)]^{\mathcal{G}}_*$,
4. $\langle R\rangle^{\mathcal{G}}_{\alpha} = \langle R{\ni}{\in}\rangle^{\mathcal{G}}_* = \langle \delta_i(R)\rangle^{\mathcal{G}}_*$ *and* $[R]^{\mathcal{G}}_{\alpha} = [R{\ni}{\in}]^{\mathcal{G}}_* = [\delta_i(R)]^{\mathcal{G}}_*$,
5. $\langle S\rangle^{\mathcal{G}}_r = [\Lambda(S)]^{\mathcal{G}}_*$ *and* $[S]^{\mathcal{G}}_r = \langle \Lambda(S)\rangle^{\mathcal{G}}_*$.
6. $\langle R\rangle^{\mathcal{G}}_{\alpha} = [\delta_o(R)]^{\mathcal{G}}_*$ *and* $[R]^{\mathcal{G}}_{\alpha} = \langle \delta_o(R)\rangle^{\mathcal{G}}_*$.

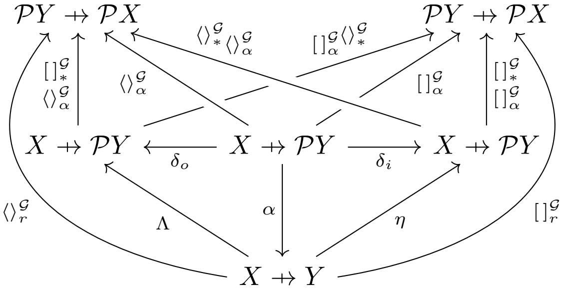
Proof. For (1), $\langle \Lambda(S)\rangle^{\mathcal{G}}_{\alpha} = \langle \alpha(\Lambda(S))\rangle^{\mathcal{G}}_r = \langle S\rangle^{\mathcal{G}}_r$. The proofs for η and the box operators are similar.

For (3), $\Lambda((S{\in}){\uparrow}^{\smile}) = \Lambda((S{\in}\Omega)^{\smile}) = \Lambda((S{\in})^{\smile}) = \mathcal{P}(S^{\smile})$ using ${\in}\Omega = {\in}$. A proof of the second identity uses $\eta(S){\uparrow} = S1\Omega = S{\in}$ and is similar; and so are the remaining proofs.

For (5), $\Lambda(\Lambda(S){\uparrow}^{\smile}) = \Lambda((\Lambda(S)\Omega)^{\smile}) = \Lambda({\ni}/S)$ using Lemma 5.10(2) shows the second identity; the first follows by duality.

Items (2), (4) and (6) follow by instantiating (1), (3) and (5) with $S = \alpha(R)$. □

The following commutative diagram summarises the relationships between the various modal operators shown in the previous results.



6 Goldblatt's Axioms for Concurrent Dynamic Logic

The box and diamond axioms of Peleg's concurrent dynamic logic have already been derived in the multirelational semantics [FS15]. In particular, the Kleene star R^* of $R : X \nrightarrow \mathcal{P}X$ has been defined as the least fixpoint of $\lambda S.\ 1 \cup R * S$ and studied in this setting. It exists because this function preserves the order $\subseteq$ of the complete lattice $X \nrightarrow \mathcal{P}X$. While the general algebra of multirelations does not satisfy the usual star axioms of Kleene algebra (owing for instance to the absence of associativity of Peleg composition), the modal star axioms of concurrent dynamic algebra can nevertheless be derived. Relative to these results it remains to derive Goldblatt's box axioms, which consider $[-]_\alpha$ in combination with $\langle - \rangle_*$, and disregard $[-]_*$. To simplify notation, we write $\langle - \rangle$ for $\langle - \rangle_*$ and $[-]$ for $[-]_\alpha$ in the following.

In algebraic form, Goldblatt's axioms [Gol92] for concurrent dynamic logic are

$$[R](P \to Q) \subseteq ([R]P \to [R]Q), \tag{G1}$$
$$[R]1 = 1, \tag{G2}$$
$$[R * S]P = [R][S]P, \tag{G3}$$
$$[R \cup S]P = [R]P \cap [S]P, \tag{G4}$$
$$[R \Cup S]P = (\langle R \rangle 1 \to [S]P) \cap (\langle S \rangle 1 \to [R]P), \tag{G5}$$
$$[R^*]P \subseteq P \cap [R][R^*]P, \tag{G6}$$
$$[R^*](P \to [R]P) \subseteq (P \to [R^*]P), \tag{G7}$$
$$[P]Q = P \to Q, \tag{G8}$$
$$[R](P \to Q) \subseteq \langle R \rangle P \to \langle R \rangle Q, \tag{G9}$$
$$\langle R * S \rangle P = \langle R \rangle \langle S \rangle P, \tag{G10}$$
$$\langle R \cup S \rangle P = \langle R \rangle P \cup \langle S \rangle P, \tag{G11}$$
$$\langle R \Cup S \rangle P = \langle R \rangle P \cap \langle S \rangle P, \tag{G12}$$
$$P \cup \langle R \rangle \langle R^* \rangle P = \langle R^* \rangle P, \tag{G13}$$
$$[R^*](\langle R \rangle P \to P) \subseteq (\langle R^* \rangle P \to P), \tag{G14}$$
$$\langle P \rangle Q = P \cap Q, \tag{G15}$$
$$[R]\emptyset \cup \langle R \rangle 1 = 1, \tag{G16}$$

where P, Q are tests and $P \to Q$ stands for $\neg P \cup Q$.

Theorem 6.1. *All axioms except (G3) hold in the multirelational semantics.*

Proof. Axioms (G10)-(G13) and (G15) belong to Peleg's concurrent dynamic logic. Algebraic variants have been derived [FS15]. The remaining axioms, except (G3), have been validated with the Isabelle/HOL proof assistant.

As a counterexample to (G3), consider the multirelation $R = \{(a, \{a, b\})\}$ on $\{a, b\}$ from Example 3.3 with $[R * R]\emptyset$. Then $\alpha(R * R) = \emptyset \subset \{(a, a), (a, b)\} = \alpha(R)\alpha(R)$ and therefore

$$[R][R]\emptyset = [\alpha(R)\alpha(R)]\emptyset = \{(b, b)\} \subset \{(a, a), (b, b)\} = [\emptyset]\emptyset = [R * R]\emptyset. \qquad \square$$

Goldblatt's original axioms are therefore unsound with respect to the intended multirelational semantics used by Peleg, Nerode and Wijesekera as well as by Goldblatt himself and in this article. The failure of (G3) may seem surprising: after all, $(R * S) * P = R * (S * P)$ holds for all composable multirelations R and S and multirelational tests P [FS15]. Yet the proof of Theorem 6.1 shows that the weak preservation of Peleg composition by α, namely

$$\alpha(R * S) \subseteq \alpha(R)\alpha(S),$$

in Lemma 3.2 together with Example 3.3 blocks any proof of (G3). In light of Lemma 3.2, at least a weak version of (G3) can be derived.

Proposition 6.2. *Let $R : X \nrightarrow \mathcal{P}Y$, $S : Y \nrightarrow \mathcal{P}Z$ and $P \subseteq Z$. Then $[R][S]P \subseteq [R * S]P$.*

Proof. $[R]_\alpha[S]_\alpha P = [\alpha(R)]_r[\alpha(S)]_r P = [\alpha(R)\alpha(S)]_r P \subseteq [\alpha(R * S)]_r P = [R * S]_\alpha P$, because boxes are order-reversing in their first arguments. □

Equality holds only in special cases such as $[R{\downarrow} * S]P = [R][S]P$ or if S is outer total. The inclusion from Proposition 6.2 should therefore replace (G3) in Goldblatt's axioms.

Goldblatt proves that his extension of propositional dynamic logic, which is a variant of Peleg's concurrent dynamic logic [Pel87] with the Nerode-Wijesekera box operator, is finitely axiomatisable and has the finite model property, which implies decidability. Whether these results still hold for the sound axiomatisation using the formula in Proposition 6.2 remains to be seen.

Remark 6.3. The above counterexample also shows that $[-]_\alpha$, and consequently $\langle - \rangle_\alpha$, do not generally yield actions of multirelations on powersets with respect to $*$, which is atypical for (multi)modal operators; see for instance the axioms of propositional dynamic logic. For relational modalities,

$$[RS]_r = [R]_r \circ [S]_r \qquad \text{and} \qquad \langle RS \rangle_r = \langle R \rangle_r \circ \langle S \rangle_r;$$

in fact, $[-]_r$ and $\langle - \rangle_r$ are functors $\mathbf{Rel} \to \mathbf{Set}$.

Similarly, for Peleg's multirelational modalities,

$$\langle R * S \rangle_* = \langle R \rangle_* \circ \langle S \rangle_* \qquad \text{and} \qquad [R * S]_* = [R]_* \circ [S]_*,$$

so that $[-]_*$ and $\langle - \rangle_*$ are composition-preserving maps from the algebra of multirelations (which is not a category) into **Set**.

For $\langle - \rangle_\alpha$ and $[-]_\alpha$, by contrast, these relationships fail due to the approximative nature of α, as shown in Theorem 6.1 and Proposition 6.2.

At least in the outer and inner deterministic case, we get similar results, owing to the isomorphisms with **Rel**. In the outer deterministic case, for $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$,

$$[\Lambda(R) * \Lambda(S)]_\alpha = [\Lambda(R)]_\alpha \circ [\Lambda(S)]_\alpha, \qquad \langle \Lambda(R) * \Lambda(S) \rangle_\alpha = \langle \Lambda(R) \rangle_\alpha \circ \langle \Lambda(S) \rangle_\alpha,$$
$$[\Lambda(Id)]_\alpha = Id = \langle \Lambda(Id) \rangle_\alpha.$$

Likewise, in the inner deterministic case,

$$[\eta(R) * \eta(S)]_\alpha = [\eta(R)]_\alpha \circ [\eta(S)]_\alpha, \qquad \langle \eta(R) * \eta(S) \rangle_\alpha = \langle \eta(R) \rangle_\alpha \circ \langle \eta(S) \rangle_\alpha,$$
$$[\eta(Id)]_\alpha = Id = \langle \eta(Id) \rangle_\alpha.$$

7 Propositional Hoare Logic for Multirelational Programs

The laws of propositional Hoare logic – disregarding assignment laws – can be derived in standard propositional dynamic logic. A propositional Hoare logic for multirelational programs with Peleg composition as sequential composition and inner union as parallel composition, has been proved sound with respect to a multirelational semantics in [Str18]. The multirelational semantics of conditionals and while loops has been encoded, for any test $P : X \nrightarrow X$ and multirelation $R : X \nrightarrow X$, as

$$\textbf{if } P \textbf{ then } R \textbf{ else } S = P * R \cup \neg P * S \qquad \text{and} \qquad \textbf{while } P \textbf{ do } R = (P * R)^* * \neg P.$$

Yet the validity of Hoare triples has not been encoded using box operators, as it seems unlikely, in light of Nerode and Wijesekera's argument, that Peleg's boxes capture the intended semantics.

In this section we base the validity of Hoare triples on Nerode and Wijesekera's box,

$$\vdash \{P\}R\{Q\} \Leftrightarrow P \subseteq [R]_\alpha Q,$$

for precondition $P : X \nrightarrow X$, multirelational program $R : X \nrightarrow X$ and postcondition $Q : X \nrightarrow X$. We prove soundness of the standard rules of propositional Hoare logic, plus a new rule for parallel composition, with respect to the multirelational semantics.

Proposition 7.1. *The following rules of a propositional Hoare logic for multirelational programs are derivable in the multirelational semantics:*

$$
\begin{aligned}
P &\subseteq [1]_\alpha P,\\
P' \subseteq P \wedge P \subseteq [R]_\alpha Q \wedge Q \subseteq Q' &\Rightarrow P' \subseteq [R]_\alpha Q',\\
P \subseteq [R]_\alpha P' \wedge P' \subseteq [S]_\alpha Q &\Rightarrow P \subseteq [R * S]_\alpha Q,\\
P \cap Q \subseteq [R]_\alpha Q' \wedge \neg P \cap Q \subseteq [S]_\alpha Q' &\Rightarrow Q \subseteq [\mathbf{if}\ P\ \mathbf{then}\ R\ \mathbf{else}\ S]_\alpha Q',\\
P \cap Q \subseteq [R]_\alpha Q &\Rightarrow Q \subseteq [\mathbf{while}\ P\ \mathbf{do}\ R]_\alpha (Q - P),\\
P \subseteq [R]_\alpha Q \wedge P \subseteq [S]_\alpha Q &\Rightarrow P \subseteq [R \Cup S]_\alpha Q.
\end{aligned}
$$

Proof. The skip and consequence rules for multirelations are immediate from their relational counterparts, using $[-]_\alpha = [\alpha(-)]_r$.

For the sequential-composition rule, the hypotheses for this rule, standard laws for relational boxes and the repaired axiom (G3), $[R]_\alpha [S]_\alpha P \subseteq [R * S]_\alpha P$, from Proposition 6.2 imply that

$$P \subseteq [\alpha(R)]_r P' \subseteq [\alpha(R)]_r [\alpha(S)]_r Q = [R]_\alpha [S]_\alpha Q \subseteq [R * S]_\alpha Q.$$

For the conditional rule, $Q \subseteq [P]_\alpha (P \cap Q)$ holds by boolean algebra and (G8). The hypotheses for this rule and the sequential-composition rule then yield $Q \subseteq [P * R]_\alpha Q'$ and $Q \subseteq [\neg P * S]_\alpha Q'$, and $Q \subseteq [P * R \cup \neg P * S]_\alpha Q'$ follows from (G4).

For the while rule, first note that $P \subseteq [R]_\alpha P \Rightarrow P \subseteq [R^*]_\alpha P$ for arbitrary R, P: the antecedent is equivalent to $1 = P \to [R]_\alpha P$ and $1 = [R^*]_\alpha 1 = [R^*]_\alpha (P \to [R]_\alpha P) = P \to [R^*]_\alpha P$, using (G2) and (G7), implies the consequent. Further, as for conditionals, $Q \subseteq [P * R]_\alpha Q$ follows from the hypothesis of the while rule and the sequential-composition rule; so we get $Q \subseteq [(P * R)^*]_\alpha Q$ from this implication. Using sequential composition, we combine this inequality with $Q \subseteq [\neg P]_\alpha (Q - P)$, an immediate consequence of (G8), into $Q \subseteq [(P * R)^* * \neg P]_\alpha (Q - P)$.

Lastly, for the parallel-composition rule, $P \subseteq (\langle R\rangle_* 1 \to [S]_\alpha Q) \cap (\langle S\rangle_* 1 \to [R]_\alpha Q) = [R \Cup S]_\alpha Q$ follows from the hypotheses of this rule using (G5) and some boolean algebra. □

In addition to this proof we have also derived the rules in Proposition 7.1 using Isabelle. In conclusion, despite its weaknesses discussed in Section 6, Nerode and Wijesekera's box operator seems nevertheless useful for program correctness and verification.

8 Conclusion

In this trilogy of articles we have studied the inner structure of multirelations and the categories of outer and inner deterministic and univalent multirelations. Here, in the final part of this trilogy, we have used this development to formalise Nerode and Wijesekera's alternative box operator for concurrent dynamic logic in an extension of power allegories, added a new De Morgan dual diamond and related these two operators with Peleg's modalities for multirelations and with relational modalities in **Set** and **Rel**, answering a question from [FS15]. As applications, we have derived an algebraic variant of Goldblatt's axioms for concurrent dynamic logic and shown that Nerode and Wijesekera's box does not yield an action on sets, which is atypical for (multi)modal operators. We have also shown how this operator can be used as the basis of a Hoare logic for multirelational programs, which may be useful in program verification.

While we use a multirelational language of concrete relations and multirelations in this work, an axiomatic extension of the abstract allegorical approach, which equips boolean power allegories with multirelational operations, is its most natural continuation. The characterisation of intuitionistic modal algebras based on locally complete allegories is another interesting question. Beyond the forward modal operators considered so far, backward modalities could be defined using our structural approach, and their application in the semantics and verification of programs or specifications with alternating nondeterminism should be explored. The use of concurrent dynamic logic in (multirelational) program analysis is an interesting perspective for applications. Our derivation of a propositional Hoare logic for multirelational programs provides new evidence in this regard. A general approach for transforming algebraic formalisms like the one in this article into program verification components with proof assistants, including models of the program store and for variable assignments, has been described in [AGS16]. The inner preorders for multirelations defined in [FGS23b] seem related to the testing preorders for probabilistic processes studied by Deng et al. [DvGHM08]. It might therefore be interesting to see how the modalities from concurrent dynamic logics relate to the Hennessy-Milner modalities in their approach.

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References

[AGS16] A. Armstrong, V. B. F. Gomes, and G. Struth. Building program construction and verification tools from algebraic principles. *Formal Aspects of Computing*, 28(2):265–293, 2016.

[BdM97] R. Bird and O. de Moor. *Algebra of Programming.* Prentice Hall, 1997.

[DvGHM08] Y. Deng, R. van Glabbeek, M. Hennessy, and C. Morgan. Characterising testing preorders for finite probabilistic processes. *Logical Methods in Computer Science*, 4(4):1–33, 2008.

[FGS23a] H. Furusawa, W. Guttmann, and G. Struth. Determinism of multirelations. *arXiv*, 2305.11344 version 2, 2023. https://arxiv.org/abs/2305.11344v2.

[FGS23b] H. Furusawa, W. Guttmann, and G. Struth. On the inner structure of multirelations. *arXiv*, 2305.11342 version 2, 2023. https://arxiv.org/abs/2305.11342v2.

[FKST17] H. Furusawa, Y. Kawahara, G. Struth, and N. Tsumagari. Kleisli, Parikh and Peleg compositions and liftings for multirelations. *Journal of Logical and Algebraic Methods in Programming*, 90:84–101, 2017.

[FŠ90] P. J. Freyd and A. Ščedrov. *Categories, Allegories*, volume 39 of *North-Holland Mathematical Library.* Elsevier Science Publishers, 1990.

[FS15] H. Furusawa and G. Struth. Concurrent dynamic algebra. *ACM Transactions on Computational Logic*, 16(4:30):1–38, 2015.

[FS16] H. Furusawa and G. Struth. Taming multirelations. *ACM Transactions on Computational Logic*, 17(4:28):1–34, 2016.

[Gol92] R. Goldblatt. Parallel action: Concurrent dynamic logic with independent modalities. *Studia Logica*, 51(3–4):551–578, 1992.

[GS23] W. Guttmann and G. Struth. Inner structure, determinism and modal algebra of multirelations. *Archive of Formal Proofs*, 2023. Formal proof development, https://isa-afp.org/entries/Multirelations_Heterogeneous.html.

[NW90] A. Nerode and D. Wijesekera. Constructive concurrent dynamic logic I. Technical Report Mathematical Sciences Institute 90-43, Cornell University, 1990.

[Pel87] D. Peleg. Concurrent dynamic logic. *Journal of the ACM*, 34(2):450–479, 1987.

[Rew03] I. Rewitzky. Binary multirelations. In H. de Swart, E. Orłowska, G. Schmidt, and M. Roubens, editors, *Theory and Applications of Relational Structures as Knowledge Instruments*, volume 2929 of *Lecture Notes in Computer Science*, pages 256–271. Springer, 2003.

[Sch11] G. Schmidt. *Relational Mathematics*. Cambridge University Press, 2011.

[SS89] G. Schmidt and T. Ströhlein. *Relationen und Graphen*. Springer, 1989.

[Str18] G. Struth. Hoare semigroups. *Mathematical Structures in Computer Science*, 28(6):775–799, 2018.

A Basis

Almost every operation in this paper, as well as [FGS23b, FGS23a], can be defined in terms of a basis of six operations that mix the relational and the multirelational language: the relational operations $-$, $\cap$, $/$ and the multirelational operations 1, $\Cup$, $*$. Here we extend the list from [FGS23a] with definitions of the modal operators (all operations listed feature in our trilogy of articles, but not all of them in the present paper).

- $R \cup S = -(-R \cap -S)$
- $R - S = R \cap -S$
- $\emptyset = R \cap -R$
- $U = -\emptyset$
- $R\!\uparrow = R \Cup U$
- $\in \, = 1\!\uparrow$
- $Id = 1/1$
- $R^{\smile} = -(-Id/R)$
- $SR = -(-S/R^{\smile})$
- $\ni \, = \, \in^{\smile}$
- $R\backslash S = (S^{\smile}/R^{\smile})^{\smile}$
- $R \div S = (R\backslash S) \cap (R^{\smile}/S^{\smile})$
- $\Lambda(R) = R^{\smile} \div \in$
- $\mathcal{P}(R) = \Lambda(\ni R)$
- $R_{\mathcal{P}} = \mathcal{P}(R\ni)$
- $\mu = Id_{\mathcal{P}}$
- $\Omega = \in\backslash\in$
- $C = \in \div - \in$
- $\sim\! R = RC$
- $R \Cap S = \sim\!(\sim\! R \Cup \sim\! S)$
- $R\!\downarrow = X \Cap U$
- $R\!\updownarrow = R\!\uparrow \cap R\!\downarrow$
- $1_{\Cup} = 1 \Cap \sim\! 1$
- $1_{\Cap} = \sim\! 1_{\Cup}$
- $R^{\mathsf{d}} = -\!\sim\! R$
- $R \odot S = \sim\!(R * \sim\! S)$
- $R_* = (\Lambda(\ni 1) * 1^{\smile} R1)\mu$
- $R/S = R/S_*$
- $\mathsf{A}_{\Cup} = U1$
- $\mathsf{A}_{\Cap} = \sim\!\mathsf{A}_{\Cup}$
- $\nu(R) = R - 1_{\Cup}$
- $\tau(R) = R \cap 1_{\Cup}$
- $\alpha(R) = R\ni$
- $\delta_i(R) = R\!\downarrow \cap \mathsf{A}_{\Cup}$
- $\delta_o(R) = 1R_{\mathcal{P}}$
- $\tilde{\delta_i}(R) = R\!\uparrow \cap \mathsf{A}_{\Cap}$
- $\tilde{\delta_o}(R) = \sim\!\delta_o(\sim\! R)$
- $dom(R) = Id \cap RR^{\smile}$
- $\neg P = Id - P$
- $\langle R\rangle_r P = dom(RP)$
- $[R]_r P = \neg\langle R\rangle_r \neg P$
- $\langle R\rangle_* P = dom(R * P)$
- $[R]_* P = \neg\langle R\rangle_* \neg P$
- $\langle R\rangle_\alpha P = \langle\alpha(R)\rangle_r P$
- $[R]_\alpha P = \neg\langle R\rangle_\alpha \neg P$
- $\langle R\rangle_r^{\mathcal{G}} = \mathcal{P}(R^{\smile})$
- $[R]_r^{\mathcal{G}} = \Lambda(\ni/R)$
- $\langle R\rangle_*^{\mathcal{G}} = \Lambda(R\!\uparrow^{\smile})$
- $[R]_*^{\mathcal{G}} = \Lambda(R\!\uparrow^{\mathsf{d}\smile})$
- $\langle R\rangle_\alpha^{\mathcal{G}} = \langle\alpha(R)\rangle_r^{\mathcal{G}}$
- $[R]_\alpha^{\mathcal{G}} = [\alpha(R)]_r^{\mathcal{G}}$
- $R \sqsubseteq_{\uparrow} S \Leftrightarrow S \subseteq R\!\uparrow$
- $R \sqsubseteq_{\downarrow} S \Leftrightarrow R \subseteq S\!\downarrow$
- $R \sqsubseteq_{\updownarrow} S \Leftrightarrow R \sqsubseteq_{\downarrow} S \wedge R \sqsubseteq_{\uparrow} S$

If $*$ is extended to relations, the simpler definition $R_* = Id * R$ may be used. Alternatively, we could of course replace Peleg composition by Peleg lifting in the basis. We could also replace relational intersection $\cap$ with a multirelational intersection variant $\cap$ in the basis: relational $\cap$ is obtained by $R \cap S = \alpha(R1 \cap S1)$ which can be defined in terms of multirelational $\cap$ and the rest of the basis. Yet we do not know whether a multirelational $-$ could replace the relational variant. Finally, relational $/$ is required to define some of the operations in our list as it is the only operation in the basis that can change types. We have so far not attempted to axiomatise the basic operations in the sense of (heterogeneous) relation algebra [SS89], concurrent dynamic algebra [FS16] or likewise.