Determinism of Multirelations

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Abstract

Binary multirelations allow modelling alternating nondeterminism, for instance, in games or nondeterministically evolving systems interacting with an environment. Such systems can show partial or total functional behaviour at both levels of alternation, so that nondeterministic behaviour may occur only at one level or both levels, or not at all. We study classes of inner and outer partial and total functional multirelations in a multirelational language based on relation algebra and power allegories. While it is known that general multirelations do not form a category, we show in the multirelational language that the classes of deterministic multirelations mentioned form categories with respect to Peleg composition from concurrent dynamic logic, and sometimes quantaloids. Some of these categories are isomorphic to the category of binary relations. We also introduce determinisation maps that approximate multirelations either by binary relations or by deterministic multirelations. Such maps are useful for defining modal operators on multirelations.

Keywords: binary relations, binary multirelations, categories, deterministic multirelations, power allegories, quantaloids

1. Introduction

This is the second article in a trilogy on the inner structure of multirelations [5], their determinisation and their algebras of modal operators [4].

Multirelations are binary relations of type $X \nrightarrow \mathcal{P}Y$, which we study in the category **Rel** with sets as objects and binary relations $X \nrightarrow Y$ as arrows. As explained in [5], they form models of alternating angelic and demonic nondeterminism, while arbitrary relations are standard models of angelic nondeterminism without alternation. At the outer or angelic level of nondeterminism, each element in X can be related by a multirelation to one subset or many subsets of Y, or to no set at all. At the inner or demonic level of nondeterminism, it can be related within each of these subsets to one element or many elements, or to no element at all. Multirelational semantics of programs with angelic and demonic nondeterminism have been proposed by Rewitzky [17]. Multirelations also feature implicitly, for instance, in the semantics of Parikh's game logics [14], in Peleg's concurrent dynamic logic [15] and in the transition relations of alternating automata.

In the first part of this trilogy we have studied the inner or demonic structure of multirelations, which complements the usual angelic boolean structure on relations. A typical inner operation is the inner union of two multirelations R, S of the same type: if R and S relate an element a with the sets B and C, respectively, then $R \Cup S$ relates a with $B \cup C$. Operations of inner intersection and inner complementation can then be defined in the obvious way, performing set-intersection or set-complementation on the second components of ordered pairs.

We have also discussed notions of inner univalence or inner partial functionality, inner totality and inner determinism or functionality, which complement the standard outer notions from relation

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algebra. A multirelation is inner univalent if every element in its codomain is either empty or a singleton set. Relating to the empty set thus represents inner partiality. A multirelation is therefore inner total if every element in its domain is related to a non-empty set, and inner deterministic if it is inner univalent and inner total. By contrast, an outer univalent multirelation is (the graph of) a partial function, an outer total multirelation relates every element with some set, including the empty one, and an outer deterministic multirelation is (the graph of) a function. Intuitively, inner univalent multirelations can thus be seen as angelic multirelations that do not allow any inner or demonic choices, while outer univalent multirelations can be seen as demonic, as they do not allow any outer or angelic choices [17]. Inner deterministic multirelations are therefore strictly angelic, as empty inner choices are forbidden. Outer deterministic multirelations are strictly demonic, as empty outer choices are impossible.

The multirelation $R = \{(a, \emptyset), (b, \{c\}), (b, X)\}$ on the set $X = \{a, b, c\}$, for example, is neither outer nor inner univalent. It is not outer univalent because b is related to two different sets and not inner univalent because b is related to a set with three different elements. It is not outer total because c is not related to any element and not inner total because a is related to the empty set. The multirelations $S = R - \{(b, \{c\})\}$ and $S \cup \{(c, X)\}$ are outer univalent and outer deterministic, respectively. The multirelations $T = R - \{(b, X)\}$ and $T - \{(a, \emptyset)\}$ are inner univalent and inner deterministic, respectively.

Here we study the structure of inner and outer univalent and deterministic multirelations in an algebraic multirelational language [6] that combines features of relation algebra [19], quantaloids [16, 18] and power allegories [2, 3] with specific operations for multirelations. We also consider the determinisation of multirelations by relations or by deterministic multirelations.

Apart from the operations on the inner and outer structure mentioned, we consider the Peleg composition of multirelations [15], which has been introduced in the context of concurrent dynamic logic and is one of several possible compositions for multirelations. See [5, 7] for a discussion of this operation, its relevance to computing and examples. Multirelations under Peleg composition do not form categories because this operation is not associative. Yet specific classes of multirelations do, for instance, the classes of deterministic or univalent multirelations (Proposition 2.8), which follows from results in [6]. In Propositions 3.2 and 5.6 we show that inner deterministic and inner univalent multirelations form categories with respect to Peleg composition as well. In particular, the power transpose map from power allegories is in fact a functor from **Rel** to categories of inner deterministic multirelations with respect to Peleg composition (Lemma 3.5). Moreover, the categories of inner and outer deterministic multirelations are isomorphic to the category **Rel** of sets and relations (Proposition 3.6) and its enrichment in the form of quantaloids (Proposition 3.10).

We further introduce maps that approximate multirelations by relations or their isomorphic inner or outer deterministic multirelations. These determinisation maps are related to the original multirelation by Galois connections with respect to one of the inner preorders, which compare the inner nondeterminism of multirelations (Proposition 4.3). They are also functors between the categories of inner and outer deterministic multirelations (Corollary 4.1), and the inner and outer deterministic multirelations arise as their fixpoints. One of the determinisation maps is further used in the proof that inner univalent multirelations form a category (Proposition 5.6).

Apart from these main results, we present further structural insights, for instance, refinements of the properties of inner or outer deterministic multirelations to inner or outer univalent or total ones in Section 6 or an outline of properties of dual determinisation maps in Section 7. We also present a number of calculational properties which belong in the conceptual context of this article, but are only needed in the third part of our trilogy.

As in [5], we work in concrete extensions and enrichments of **Rel**, but with a view towards future axiomatic approaches to our multirelational language. We therefore aim at algebraic proofs if possible, but nevertheless present set-theoretic definitions of all important concepts. While many properties in this article have simple set-theoretic proofs, proofs involving Peleg composition can be surprisingly tedious and may involve a variant of the Axiom of Choice. Algebraic reasoning often circumvents this complexity, which is another reason for emphasising it here. Beyond that, as everywhere in this trilogy, we have used the Isabelle/HOL proof assistant to check most of our results, in particular the more tedious ones. We have developed a substantial mathematical component for multirelations [10], and concrete power allegories more generally. Yet we did not aim at a complete formalisation and our article is self-contained without these components.

2. Relations and Multirelations

We first recall the basics of binary relations and multirelations; see [5] and the references therein for details. Our algebraic language of concrete relations and multirelations is based on enrichments of the category **Rel** with sets as objects, relations as arrows and the identity or diagonal relations as identity arrows. Yet in contrast to [5] we now extend the standard calculus of relations [19] with concepts from power allegories [3] using in particular their connection with the monad of the powerset functor in the category **Set** of sets and functions [2], and with specific multirelational concepts from [6]. The richness of this language sometimes prevents us from listing all properties used in calculations and proofs – we then refer to "standard" properties. We extend the dependency list of relational and multirelational concepts of our multirelational language with respect to a small basis from [5] accordingly in Appendix A.

2.1. Binary relations

The category **Rel** forms a quantaloid [18]: each of its homsets forms a complete lattice and arrow composition preserves arbitrary sups in both arguments. Relational composition thus has two residuals as right adjoints. The homsets of **Rel** form quantales based on complete atomic boolean algebras. Moreover, relational converse is an involution and a contravariant endofunctor of quantaloids. We write $X \nrightarrow Y$ for the homset $\mathbf{Rel}(X,Y)$, Id_X for the identity relation on X, $\emptyset_{X,Y}$ for the least and $U_{X,Y}$ for the greatest element in $X \nrightarrow Y$, $-R$ for the complement of R and $S - R$ for the relative complement $S \cap -R$, RS for the relational composition of relations R, S of suitable types, R/S and $R\backslash S$ for the left and right residuals of R and S and $R^\smile$ for the converse of R. The modular law $RS \cap T \subseteq (R \cap TS^\smile)S$ links composition, intersection and converse in **Rel**, so that this category forms in fact a modular quantaloid [18].

We need the properties $T\backslash S = (S^\smile/T^\smile)^\smile$, $T/S = -(-TS^\smile)$ and $T\backslash S = -(T^\smile(-S))$ of residuals. We also need the following standard concepts of algebras of binary relations:

- the *symmetric quotient* $T \div S : X \nrightarrow Y$ of $T : X \nrightarrow Z$ and $S : Y \nrightarrow Z$, defined as $T \div S = (T\backslash S) \cap (T^\smile/S^\smile)$,
- *tests*, which are relations $R \subseteq Id$, and whose relational composition is intersection,
- the *domain* map $dom : (X \nrightarrow Y) \to (X \nrightarrow X)$ defined by $dom(R) = Id_X \cap RR^\smile = Id_X \cap RU_{Y,X} = \{(a,a) \mid \exists b.\ (a,b) \in R\}$.

Tests form a full subalgebra of $\mathbf{Rel}(X,X)$ for any X, which is again a complete atomic boolean algebra in which multiplication coincides with binary inf.

We are particularly interested in deterministic (multi)relations. The relation $R : X \nrightarrow Y$ is

- *total* if $dom(R) = Id_X$, or equivalently $Id_X \subseteq RR^\smile$,
- *univalent*, or a *partial function*, if $R^\smile R \subseteq Id_Y$,
- *deterministic*, or a *function*, if it is total and univalent.

Functions as deterministic relations in **Rel** are nothing but graphs of functions in **Set**. They can be used to model programs as a subclass of nondeterministic specifications in program refinement calculi. We need the equational modular law $RS \cap T = (R \cap TS^\smile)S$ for univalent S [20] in calculations.

We further write $R|_A = \{(a,b) \in R \mid a \in A\}$ for the restriction of relation R to elements in the set A, $R(A)$ for the relational image of A under R and $R(a)$ for $R(\{a\})$.

Next we recall the basic concepts of power allegories [2, 3], on which our multirelational language is based. The isomorphism between relations in $X \nrightarrow Y$ and nondeterministic functions in

$X \to \mathcal{P}Y$ in **Set** can be expressed in **Rel**. Nondeterministic functions $X \to \mathcal{P}Y$ in **Set** correspond of course to functions $X \nrightarrow \mathcal{P}Y$ in **Rel**.

The *power transpose*

$$\Lambda : (X \nrightarrow Y) \to (X \nrightarrow \mathcal{P}Y),\; R \mapsto \{(a, R(a)) \mid a \in X\}$$

maps relations $X \nrightarrow Y$ to functions in $X \nrightarrow \mathcal{P}Y$, which are graphs of nondeterministic functions $X \to \mathcal{P}Y$ in **Set**. In the other direction, relational postcomposition with the has-element relation $\ni_Y \colon \mathcal{P}Y \nrightarrow Y$, the converse of the set membership relation $\in_Y \colon Y \nrightarrow \mathcal{P}Y$, maps relations, and therefore functions in $X \nrightarrow \mathcal{P}Y$, to relations in $X \nrightarrow Y$. We henceforth write $\alpha = (-)\ni$. This function satisfies

$$\alpha : (X \nrightarrow \mathcal{P}Y) \to (X \nrightarrow Y),\; R \mapsto \left\{(a, b) \mid b \in \bigcup R(a)\right\}.$$

Algebraically, $\Lambda(R) = R^{\smile} \div \in$, and we will see below how $\ni$ and α can be expressed in terms of basic relational and multirelational operations.

The following fact from [2] summarises the relationship between Λ and α.

Lemma 2.1. *Let $R : X \nrightarrow Y$ and let $f : X \nrightarrow \mathcal{P}Y$ be deterministic. Then*

1. $f = \Lambda(R) \Leftrightarrow R = \alpha(f)$,
2. $\alpha(\Lambda(R)) = R$ *and* $\Lambda(\alpha(f)) = f$,
3. $f\Lambda(R) = \Lambda(fR)$ *and* $\Lambda(\ni_X) = Id_{\mathcal{P}X}$.

The lower triangle in the following diagram therefore commutes:

$$\begin{array}{ccc} X \nrightarrow \mathcal{P}Y & \dashrightarrow & X \nrightarrow \mathcal{P}Y \\ {\scriptstyle\alpha}\downarrow & \nearrow^{\Lambda} & \downarrow{\scriptstyle\alpha} \\ X \nrightarrow Y & \xrightarrow[id]{} & X \nrightarrow Y \end{array}$$

The upper triangle with the dotted arrow is discussed in Section 4 below. We show that it commutes if we label the dotted arrow with *id* and if the multirelations in the upper row are deterministic. In this case, Λ and α form a bijective pair.

We also need the *relational image functor* of power allegories:

$$\mathcal{P} : (X \nrightarrow Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y),\; R \mapsto \Lambda(\ni_X R).$$

Expanding definitions, $\mathcal{P}(R) = \{(A, R(A)) \mid A \subseteq X\}$, so that the relational image, given by the covariant powerset functor in **Set**, is coded again as a graph. It is deterministic by definition. As a functor, it satisfies $\mathcal{P}(RS) = \mathcal{P}(R)\mathcal{P}(S)$ and $\mathcal{P}(Id) = Id$.

The unit and multiplication of the powerset monad are recovered relationally as $\eta_X : X \nrightarrow \mathcal{P}X$ and $\mu_X : \mathcal{P}^2X \nrightarrow \mathcal{P}X$, so that $\eta_X = \Lambda(Id_X)$ and $\mu_X = \mathcal{P}(\ni_X)$. Alternatively, $\eta_X = Id_X \div \in_X$ and, expanding definitions, $\eta_X = \{(a, \{a\}) \mid a \in X\}$.

Lemma 2.2. *Let $R : X \nrightarrow Y$, $S : Y \nrightarrow Z$, let $f : X \nrightarrow Y$ be deterministic. Then*

1. $\Lambda(RS) = \Lambda(R)\mathcal{P}(S)$,
2. $\eta\mathcal{P}(R) = \Lambda(R)$ *and* $\alpha(\eta\mathcal{P}(R)) = R$*, hence* $\mathcal{P}$ *has a right inverse,*
3. $\Lambda(f) = f\eta$,
4. η *and* μ *are natural transformations:* $\eta\mathcal{P}(f) = f\eta$ *and* $\mathcal{P}^2(f)\mu = \mu\mathcal{P}(f)$,
5. *the monad axioms hold:* $\mathcal{P}(\mu)\mu = \mu\mu$, $\mathcal{P}(\eta)\mu = Id$ *and* $\eta\mu = Id$,
6. $\alpha(\eta) = Id$.

Rather unsurprisingly, $\mathcal{P}$ does not form a monad on **Rel**; it only does on its wide subcategory **Set** (up to isomorphism). In particular, property (4) above holds only for deterministic relations. This shows that the standard monadic machinery of category theory does not translate directly from **Set** to $\mathcal{P}$ in **Rel**.

The following relations are standard in relation algebra and can be defined in power allegories:

- the *subset relation* $\Omega_Y = \in_Y \backslash \in_Y = \{(A, B) \mid A \subseteq B \subseteq Y\}$,
- the *complementation relation* $C = \in_Y \div - \in_Y = \{(A, -A) \mid A \subseteq Y\}$.

We need the following fact in proofs in our multirelational language. We add algebraic proofs because it seems novel.

Lemma 2.3. *Let $R : X \nrightarrow Y$. Then*

1. $\Lambda(R)C = \Lambda(-R)$,
2. $\Lambda(R)\Omega = R^{\smile} \backslash \in = (\ni / R)^{\smile}$.

Proof. For (1), $\Lambda(R)C = \Lambda(R)\Lambda(-\ni) = \Lambda(\Lambda(R)(-\ni)) = \Lambda(-(\Lambda(R)\ni)) = \Lambda(-R)$. This uses properties of Lemma 2.1 and determinism of $\Lambda(R)$.

For (2), $\Lambda(R)\Omega = \Lambda(R)(-(\ni(-\in))) = -(\Lambda(R)\ni(-\in)) = -(R(-\in)) = R^{\smile} \backslash \in$, using the definition of Ω in the first step, determinism of $\Lambda(R)$ in the second, Lemma 2.1 in the third and properties of residuals in the fourth. □

The *power test* [6] $P_* : \mathcal{P}X \nrightarrow \mathcal{P}X$ of a test $P \subseteq Id_X$ is defined as

$$P_* = (\in_X \backslash P \in_X) \cap Id_{\mathcal{P}X} = \{(A, A) \mid \forall a \in A.\ (a, a) \in P\}.$$

We use it for defining the Peleg composition of multirelations in the following section.

Finally, for $R, S : X \nrightarrow Y$, we write $S \subseteq_d R$ if S is univalent, $dom(S) = dom(R)$ and $S \subseteq R$. This allows us to decompose any relation as $R = \bigcup_{S \subseteq_d R} S$ [5, Lemma 2.1].

Relations similar to $\subseteq_d$ appear in program refinement: $S \subseteq_d R$, in particular, means that S maximally post-refines R by eliminating all nondeterminism while keeping its domain unchanged [9, 13].

2.2. Multirelations

A *multirelation* is an arrow $X \nrightarrow \mathcal{P}Y$ in **Rel** and thus a doubly-nondeterministic function $X \to \mathcal{P}^2 Y$ in **Set**. Multirelations do not form a category: the double powerset functor does not yield a suitable monad [11]. Hence there is no associative composition with suitable units [6].

The *Peleg composition* $* : (X \nrightarrow \mathcal{P}Y) \times (Y \nrightarrow \mathcal{P}Z) \to (X \nrightarrow \mathcal{P}Z)$ of multirelations [15] can be defined in terms of the *Peleg lifting* $(-)_* : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ of multirelations, which in turn can be defined in terms of the *Kleisli lifting* $(-)_{\mathcal{P}} : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ [6]:

$$R_{\mathcal{P}} = \mathcal{P}(\alpha(R)), \qquad R_* = dom(R)_* \bigcup_{S \subseteq_d R} S_{\mathcal{P}}, \qquad R * S = RS_*.$$

The definition of R_* uses the power test $dom(R)_*$. Expanding definitions,

$$\begin{aligned} R_{\mathcal{P}} &= \left\{(A, B) \mid B = \bigcup R(A)\right\}, \\ R_* &= \left\{(A, B) \mid \exists f : X \to \mathcal{P}Y.\ f|_A \subseteq R \wedge B = \bigcup f(A)\right\}, \\ R * S &= \left\{(a, C) \mid \exists B.\ (a, B) \in R \wedge \exists f : Y \to \mathcal{P}Z.\ f|_B \subseteq S \wedge C = \bigcup f(B)\right\}. \end{aligned}$$

The Kleisli lifting is the multirelational analogue of the Kleisli lifting or Kleisli extension in the Kleisli category of the powerset monad in **Set**. Its standard definition translates to multirelations.

Lemma 2.4. *Let $R : X \nrightarrow \mathcal{P}Y$. Then $R_{\mathcal{P}} = \mathcal{P}(R)\mu$.*

It can also be seen as the relational image of the relational approximation of any multirelation using α. By definition, Kleisli liftings of multirelations are functions in **Rel**.

The units of Peleg composition are given by the multirelations η_X. Because of this, we henceforth write 1_X for η_X. The following fact is structurally interesting and helpful for calculating with univalent and deterministic multirelations (see also Lemma 2.6 below).

Lemma 2.5. *Let $R : X \nrightarrow \mathcal{P}Y$, $S : Y \nrightarrow \mathcal{P}Z$ and let $f : X \nrightarrow \mathcal{P}Y$ be a function. Then the laws $(RS_\mathcal{P})_\mathcal{P} = R_\mathcal{P} S_\mathcal{P}$, $\eta_\mathcal{P} = Id$ and $\eta f_\mathcal{P} = f$ of extension systems hold.*

The standard properties $\mathcal{P}(R) = (R\eta)_\mathcal{P}$ and $\mu = Id_\mathcal{P}$, which recover the powerset monad from its extension system, still hold for multirelations. Once again, the extension system axioms work only for deterministic multirelations – the standard arrows of the Kleisli category of the powerset functor.

The interaction of Peleg composition with the outer structure is weak; see [7] for examples. As expected, it is not associative: only $(R * S) * T \subseteq R * (S * T)$ holds. Hence $(RS_*)_*$ need not be equal to $R_* S_*$, and multirelations do not form a category under Peleg composition. The composition becomes associative if the third factor is union-closed [6]. Peleg composition also preserves arbitrary unions in its first argument.

The following simple algebraic descriptions of univalent and deterministic multirelations are useful in proofs.

Lemma 2.6 ([6])**.** *Let $R : X \nrightarrow \mathcal{P}Y$. Then*

1. *$R = dom(R) 1_X R_\mathcal{P}$ and $R_* = dom(R)_* R_\mathcal{P}$ if R is univalent,*
2. *$R = 1_X R_\mathcal{P}$ and $R_* = R_\mathcal{P}$ if R is deterministic.*

As Kleisli liftings of multirelations are functions, it follows from (1) that Peleg liftings of univalent multirelations are univalent. Alternatively, $R : X \nrightarrow \mathcal{P}Y$ is univalent if and only if, for all $S : X \nrightarrow \mathcal{P}Y$, $dom(R) = dom(S)$ and $S \subseteq R$ imply $S = R$ [6]. Thus

$$S_* = dom(S)_* \bigcup_{T \subseteq_d S} T_\mathcal{P} = \bigcup_{T \subseteq_d S} dom(S)_* T_\mathcal{P} = \bigcup_{T \subseteq_d S} dom(T)_* T_\mathcal{P} = \bigcup_{T \subseteq_d S} T_*,$$

as the T are univalent, and therefore

$$R * S = R \bigcup_{T \subseteq_d S} T_* = R\, dom(S)_* \bigcup_{T \subseteq_d S} T_\mathcal{P} = R\, dom(S)_* \bigcup_{T \subseteq_d S} \mathcal{P}(\alpha(T)).$$

Univalent multirelations have stronger algebraic properties.

Lemma 2.7 ([6])**.** *Let R, S and f be composable multirelations such that f is univalent. Then*

1. *the laws $(Sf_*)_* = S_* f_*$, $\eta_* = Id$ and $\eta R_* = R$ of extension systems hold,*
2. *$(R * S) * f = R * (S * f)$.*

The following proposition follows immediately.

Proposition 2.8. *The univalent multirelations and the deterministic multirelations form categories with respect to Peleg composition and the 1_X.*

Proof. Lemma 2.7 shows that Peleg composition of univalent and therefore deterministic multirelations is associative. The 1_X are deterministic and hence univalent. It remains to show that $*$ preserves univalence and determinism. If R and S are composable univalent multirelations, then $R * S = RS_*$ is univalent because S_* is univalent and relational composition preserves univalence. If R and S are also total, then $R * S = RS_\mathcal{P}$ is total, because $S_\mathcal{P}$ is deterministic and relational composition preserves totality. □

Proposition 2.8 thus follows directly from results in [6]. We have merely expressed it in terms of categories, to align it with similar properties for inner deterministic and inner univalent multirelations introduced below. We henceforth write **URel** and **DRel** for the categories of univalent and deterministic multirelations.

In Remark 3.7 and Proposition 3.10 below we give an alternative, more structural proof for deterministic multirelations. Section 3.3 shows that deterministic multirelations form in fact quantaloids, and it features an analysis of related properties, including the relationship between the Kleisli lifting of multirelations and the Kleisli category of the powerset functor.

Definitions of inner univalence, inner totality and inner determinism depend on inner operations on multirelations. These have been studied in detail in [5], based on previous work [7, 8, 17]. Algebraic definitions relative to a small basis can be found in Appendix A.

For $R, S : X \nrightarrow \mathcal{P}Y$, one can define the

- *inner union* $R \Cup S = \{(a, A \cup B) \mid (a, A) \in R \wedge (a, B) \in S\}$ with unit $1_\Cup = \{(a, \emptyset) \mid a \in X\}$,
- *inner complementation* $\sim R = RC = \{(a, -A) \mid (a, A) \in R\}$,
- set of *atoms* $\mathsf{A}_\Cup = \{(a, \{b\}) \mid a \in X \wedge b \in Y\}$ in $X \nrightarrow \mathcal{P}Y$.

The *inner intersection* and its unit are then obtained as $R \Cap S = \sim(\sim R \Cup \sim S)$ and $1_\Cap = \sim 1_\Cup$. The operations $\Cup$ and $\Cap$ are associative and commutative, but need not be idempotent.

The multirelation $R : X \nrightarrow Y$ is

- *inner total* if $R \subseteq -1_\Cup$, that is, B is non-empty for each $(a, B) \in R$,
- *inner univalent* if $R \subseteq \mathsf{A}_\Cup \cup 1_\Cup$, that is, B is either a singleton or empty for each $(a, B) \in R$,
- *inner deterministic* if it is inner total and inner univalent, in which case $B \subseteq Y$ is a singleton set whenever $(a, B) \in R$ for some $a \in X$.

In the following we write *outer total*, *outer univalent* and *outer deterministic* instead of total, univalent and deterministic, respectively, to contrast these concepts with the inner ones.

Sets of inner univalent, inner total and inner deterministic multirelations can be characterised as fixpoints.

Lemma 2.9 ([5, Lemma 3.9])**.**

1. *The inner univalent multirelations are the fixpoints of* $(-) \cap (\mathsf{A}_\Cup \cup 1_\Cup)$.
2. *The inner total multirelations are the fixpoints of* $(-) - 1_\Cup$.
3. *The inner deterministic multirelations are the fixpoints of* $(-) \cap \mathsf{A}_\Cup$ *and* $(-)1^\smile 1$.

We also need the following closures and inner preorder, which compare the inner nondeterminism of multirelations. For $R : X \nrightarrow \mathcal{P}Y$,

- the *up-closure* $R{\uparrow} = R \Cup U = R\Omega = \{(a, A) \mid \exists (a, B) \in R.\ B \subseteq A\}$ and the *Smyth preorder* $R \sqsubseteq_\uparrow S \Leftrightarrow S \subseteq R{\uparrow}$ with equivalence $=_\uparrow$,
- the *down-closure* $R{\downarrow} = R \Cap U = R\Omega^\smile = \{(a, A) \mid \exists (a, B) \in R.\ A \subseteq B\}$ and the *Hoare preorder* $R \sqsubseteq_\downarrow S \Leftrightarrow R \subseteq S{\downarrow}$ with equivalence $=_\downarrow$.

The *convex closure* can then be defined as $R{\updownarrow} = R{\uparrow} \cap R{\downarrow}$, and the *Egli-Milner preorder* as $R \sqsubseteq_\updownarrow S \Leftrightarrow R \sqsubseteq_\downarrow S \wedge R \sqsubseteq_\uparrow S$ with equivalence $=_\updownarrow$. They are needed, for instance, in Lemma 3.8 below.

The up-closure and down-closure are related by inner duality. Using up-closure, $\in\ = 1{\uparrow}$. Moreover, $(1{\downarrow})_* = (1 \cup 1_\Cup)_* = \Omega^\smile$, and thus $R{\downarrow} = R * 1{\downarrow}$ for all $R : X \nrightarrow \mathcal{P}Y$. See Appendix A and [5] for context.

3. Outer and Inner Deterministic Multirelations

Relations $X \nrightarrow Y$ embed into multirelations $X \nrightarrow \mathcal{P}Y$ in two natural ways: postcomposition with 1_Y lifts the elements in Y to singleton sets in $\mathcal{P}Y$; taking the power transpose Λ represents the standard equivalent nondeterministic function as a multirelation $X \nrightarrow \mathcal{P}X$. The first of these embeddings yields an inner deterministic multirelation, the second an outer deterministic one:

$$\{(a, \{b\}) \mid (a, b) \in R\} \xleftarrow{(-)1_Y} R \xmapsto{\ \Lambda\ } \{(a, R(a)) \mid a \in X\}.$$

These embeddings extend to isomorphisms between the categories **Rel**, categories of inner deterministic multirelations with Kleisli composition as arrow compositions and identity arrows 1_X, and categories of outer deterministic multirelations with the same composition and identity arrows. As 1_X is the unit of the powerset monad in relational form, we henceforth write $\eta_X = (-)1_X$.

The functions Λ, α and η become functors in this setting. The isomorphism between **Rel** and the category of outer deterministic multirelations is just that between **Rel** and the Kleisli category of the powerset monad in relational form. That between **Rel** and the category of inner deterministic multirelations is trivial. A formal proof in our relational language however, requires some work. But first we check that inner deterministic multirelations form a category.

3.1. The category of inner deterministic multirelations

We start with a technical lemma.

Lemma 3.1. *Let R, S, T be composable multirelations and R inner deterministic. Then*

1. $R * S = R1^{\smile}S$,
2. $R * (S * T) = (R * S) * T$,
3. $R * S = (R * S)1^{\smile}1$ *if S is inner deterministic,*
4. $\alpha(R) = R1^{\smile}$.

Proof. For proofs of (1) and (2) see [5]. For (3), $R * S = R * S1^{\smile}1 = R1^{\smile}S1^{\smile}1 = (R * S)1^{\smile}1$, using Lemma 2.9(3) and (1). For (4), $\alpha(R) = R1^{\smile}1\ni = R1^{\smile}\alpha(1) = R1^{\smile}$ using Lemmas 2.9(3) and 2.2. □

The proofs of (1) and (2) are as simple as those of (3) and (4). The crucial insight is that inner deterministic multirelations are fixpoints of $(-)1^{\smile}1$ (Lemma 2.9(3)). The following fact is then straightforward.

Proposition 3.2. *The inner deterministic multirelations form a category with respect to Peleg composition and the 1_X.*

Proof. Lemma 3.1(2) shows that the Peleg composition of composable inner deterministic multirelations is associative. The 1_X are inner deterministic by Lemma 2.9(3) since $1 \subseteq U1 = \mathsf{A}_{\uplus}$. Lemmas 3.1(3) and 2.9(3) imply that Peleg composition preserves inner determinism. □

We henceforth write **IDRel** for this category. We present an alternative, more structural proof of Proposition 3.2 in the following section.

3.2. Isomorphisms between **Rel**, **DRel** *and* **IDRel**

Next we study the bijections between arrows in **Rel**, **DRel** and **IDRel** in detail. The results for outer deterministic multirelations are known; those for inner deterministic ones are new.

Lemma 3.3. *For every $R : X \nrightarrow Y$, $\Lambda(R)$ is outer deterministic and $\eta(R)$ inner deterministic.*

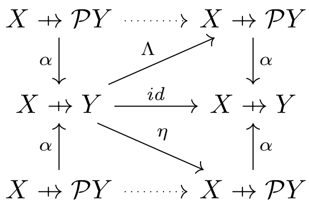
Proof. This is well known for outer determinism [2]. For inner determinism, $\eta(R) = R1 \subseteq U1 = \mathsf{A}_{\uplus}$, since relational composition preserves the order. □

Recall from Lemma 2.1 that $\alpha \circ \Lambda = id_{X \nrightarrow Y}$, while $\Lambda \circ \alpha = id_{X \nrightarrow \mathcal{P}Y}$ holds for outer deterministic multirelations. Hence Λ and α form a bijective pair between arrows in **Rel** and **DRel**. A similar fact holds for η and α.

Lemma 3.4. *The functions α and η form a bijective pair between arrows in* **Rel** *and* **IDRel**.

Proof. We need to check $\alpha \circ \eta_Y = id_{X \nrightarrow Y}$ on relations and $\eta_Y \circ \alpha = id_{X \nrightarrow \mathcal{P}Y}$ on inner deterministic multirelations. For the first identity, $\alpha(\eta(R)) = \alpha(R1) = R\alpha(1) = R\,Id = R$ using Lemma 2.2. For the second one, $\eta(\alpha(R)) = R1^{\smile}1 = R$, using Lemma 2.9(3) and Lemma 3.1(4). □

The diagram from Section 2.1 can thus be expanded:



Now the two triangles formed by solid arrows commute. The dotted arrows are discussed in Section 4. Labelling the lower dotted arrow with id makes the lower triangle commute if multirelations in the lower row are inner deterministic.

Lemma 3.4 implies that $S = \eta(R) \Leftrightarrow R = \alpha(S)$ holds for any $R : X \nrightarrow Y$ and inner deterministic $S : X \nrightarrow \mathcal{P}Y$.

Lemma 3.5. *The maps Λ, η and α extend to functors $\Lambda : \mathbf{Rel} \to \mathbf{DRel}$, $\eta : \mathbf{Rel} \to \mathbf{IDRel}$, $\alpha : \mathbf{DRel} \to \mathbf{Rel}$ and $\alpha : \mathbf{IDRel} \to \mathbf{Rel}$.*

Proof. The object components of these functors are identities, viewing multirelations $X \nrightarrow \mathcal{P}Y$ in **Rel** Kleisli-style as arrows from X to Y. Hence we focus on arrows. For $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$, we must check that

1. $\Lambda(RS) = \Lambda(R) * \Lambda(S)$ and $\Lambda(Id_X) = 1_X$,
2. $\eta(RS) = \eta(R) * \eta(S)$ and $\eta(Id_X) = 1_X$,
3. $\alpha(R * S) = \alpha(R)\alpha(S)$ and $\alpha(1_X) = Id_X$ if R and S are inner or outer deterministic.

For (1), $\Lambda(RS) = \Lambda(R)\mathcal{P}(S) = \Lambda(R)\Lambda(S)_{\mathcal{P}} = \Lambda(R)\Lambda(S)_* = \Lambda(R) * \Lambda(S)$, where the second step holds by $\Lambda(S)_{\mathcal{P}} = \mathcal{P}(\Lambda(S))\mu = \mathcal{P}(\Lambda(S))\mathcal{P}(\ni) = \mathcal{P}(\Lambda(S)\ni) = \mathcal{P}(S)$.

For (2), $\eta(R) * \eta(S) = R1 * S1 = R1(S1)_* = R(1 * S1) = RS1 = \eta(RS)$.

For (3), suppose R, S are outer deterministic. Then $R * S$ is outer deterministic by Proposition 2.8. Hence $\alpha(R)\alpha(S) = \alpha(R * S)$ if and only if $\Lambda(\alpha(R)\alpha(S)) = R * S$ because Λ and α form a bijective pair. Using this property again with (1), $\Lambda(\alpha(R)\alpha(S)) = \Lambda(\alpha(R)) * \Lambda(\alpha(S)) = R * S$. The proof for inner determinism, α and η is similar using Proposition 3.2. □

Proposition 3.6. *The categories* **Rel**, **DRel** *and* **IDRel** *are isomorphic.*

Proof. Immediate from Lemmas 3.4 and 3.5. □

The map Λ thus extends to a fully faithful functor $\mathbf{Rel} \to \mathbf{DRel}$, η extends to a fully faithful functor $\mathbf{Rel} \to \mathbf{IDRel}$ and α extends to fully faithful functors in the other directions. The isomorphisms between **Rel**, **DRel** and **IDRel** can be expressed in the language of power allegories.

Remark 3.7. The proof of Lemma 3.5 yields an alternative proof that the inner and outer deterministic multirelations form categories. It shows that Peleg composition preserves inner and outer determinism and that the 1_X are inner and outer deterministic. It implies that the Peleg composition of composable outer deterministic multirelations R, S and T is associative:

$$\begin{aligned}(R * S) * T &= (\Lambda(\alpha(R)) * \Lambda(\alpha(S))) * \Lambda(\alpha(T))\\ &= \Lambda(\alpha(R)\alpha(S)\alpha(T))\\ &= \Lambda(\alpha(R)) * (\Lambda(\alpha(S)) * \Lambda(\alpha(T)))\\ &= R * (S * T),\end{aligned}$$

A proof of associativity for inner deterministic multirelations is similar and left to the reader.

The previous results show that Peleg composition is a faithful representation of Kleisli composition of nondeterministic functions modelled as outer deterministic multirelations, and of relational composition of relations modelled as inner deterministic multirelations. This fact has been known for outer deterministic multirelations, yet the proof in this section is more structural in terms of Λ and α. As an interesting side effect, it shows that Λ is a functor $\mathbf{Rel} \to \mathbf{DRel}$. Obviously, it could not be presented as such in textbooks such as [2, 3], as the target category **DRel** based on Peleg composition and its units has been missing. The proof for inner deterministic multirelations is entirely new.

3.3. Quantaloids of deterministic multirelations

An interesting question is how the functors α, η and Λ transport the inclusion order on multirelations and relations.

Lemma 3.8.

1. *For relations R and S, $R \subseteq S$ implies $\eta(R) \subseteq \eta(S)$ and $\Lambda(R) \sqsubseteq_{\updownarrow} \Lambda(S)$.*
2. *For multirelations R and S, $R \subseteq S$ implies $\alpha(R) \subseteq \alpha(S)$.*

Proof. We first show $\Lambda(R) \sqsubseteq_{\uparrow} \Lambda(S)$, that is, $\Lambda(S) \subseteq \Lambda(R){\uparrow} = \Lambda(R)\Omega = R^{\smile}\backslash\!\in$ (see Lemma 2.3(2)). By residuation this is equivalent to $R^{\smile}\Lambda(S) \subseteq \,\in$. Since $\Lambda(S)$ is a function, this is equivalent to $R \subseteq \Lambda(S)\ni\, = S$ using Lemma 2.1, which is the assumption. Since $\Lambda(R)$ and $\Lambda(S)$ are functions, $\Lambda(R) \sqsubseteq_{\updownarrow} \Lambda(S)$ follows because $\sqsubseteq_{\uparrow}$, $\sqsubseteq_{\downarrow}$ and $\sqsubseteq_{\updownarrow}$ coincide on outer deterministic multirelations [5, Proposition 5.8]. The remaining claims follow from standard relational properties. □

The use of the Egli-Milner preorder in (1) thus captures the fact that the relationship holds also for $\sqsubseteq_{\downarrow}$ and $\sqsubseteq_{\uparrow}$ more compactly.

Example 3.9. Consider relations $R = \emptyset$ and $S = \{(a, a)\}$ on the set $\{a\}$. Then $R \subseteq S$, but $\Lambda(R) = \{(a, \emptyset)\} \nsubseteq \{(a, \{a\})\} = \Lambda(S)$.

A deeper study of these orders seems to require bicategories and is beyond the scope of this article.

The categories **DRel** and **IDRel** are enriched. We define arbitrary inner unions by

$$\Cup_{i\in I} R_i = \left\{ \left(a, \bigcup_{i\in I} A_i\right) \,\middle|\, \forall i \in I.\ (a, A_i) \in R_i \right\}$$

to capture one of the quantaloid structures that arise. Yet note that multirelations under Peleg composition and the outer operations do not form quantaloids: Peleg composition is not associative and does not preserve the sups needed [7, 8].

Proposition 3.10. *The inner deterministic multirelations with $\bigcup$ and the outer deterministic multirelations with $\Cup$ form quantaloids isomorphic to the quantaloid of binary relations.*

Proof. For the quantaloid of inner deterministic multirelations, recall that relational composition preserves arbitrary unions. Hence so do the isomorphisms η and α between **Rel** and **IDRel**:

$$\eta\left(\bigcup_{i\in I} R_i\right) = \bigcup_{i\in I} \eta(R_i) \qquad \text{and} \qquad \alpha\left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I} \alpha(S_i),$$

if all S_i are inner deterministic. Inner determinism is therefore preserved by arbitrary unions and Peleg composition distributes over arbitrary unions of inner deterministic multirelations.

For the quantaloid of outer deterministic multirelations, the isomorphisms α and Λ between **Rel** and **DRel** satisfy

$$\Lambda\left(\bigcup_{i\in I} R_i\right) = \Cup_{i\in I} \Lambda(R_i) \qquad \text{and} \qquad \alpha\left(\Cup_{i\in I} S_i\right) = \bigcup_{i\in I} \alpha(S_i)$$

if all S_i are outer deterministic. Further, the definition of arbitrary inner unions implies that they preserve outer determinism. Hence Peleg composition distributes over arbitrary inner unions of outer deterministic multirelations. □

Remark 3.11. The inner union operation is idempotent on outer univalent and hence on outer deterministic multirelations [5, Lemma 3.6]. The order of the complete lattices in **DRel** can be defined via $R \leq S \Leftrightarrow R \Cup S = S$. It is the natural order for outer deterministic multirelations [5, Lemma 5.9].

Remark 3.12. A Kleisli composition of multirelations can be defined as $R \circ_{\mathcal{P}} S = RS_{\mathcal{P}}$ [6]. It satisfies the standard identity $R \circ_{\mathcal{P}} S = R\mathcal{P}(S)\mu$, is associative on arbitrary multirelations of appropriate type and has 1 as its right unit. Moreover, 1 is a left unit of Kleisli composition for outer deterministic multirelations. By Lemma 2.6, the Peleg and Kleisli liftings coincide on **DRel**. Finally, **DRel** is isomorphic to the Kleisli category of the powerset functor in **Set**, using the graph functor to map from $X \to \mathcal{P}Y$ to $X \nrightarrow \mathcal{P}Y$, which is clearly bijective.

4. Determinisation of Multirelations

The maps $\Lambda \circ \alpha$ and $\eta \circ \alpha$ approximate multirelations by relations modelled as isomorphic inner or outer deterministic multirelations. They also yield the isomorphism between **DRel** and **IDRel**. We study them in this section.

4.1. Determinisation maps

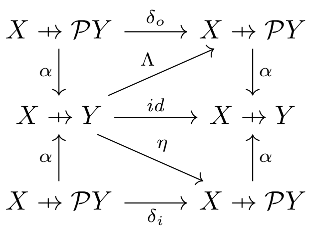
Let $R : X \nrightarrow \mathcal{P}Y$ be a multirelation. The *outer determinisation* or *fusion* map

$$\delta_o = \Lambda \circ \alpha$$

sends R to the outer deterministic multirelation isomorphic to the relation $\alpha(R)$. The *inner determinisation* or *fission* map

$$\delta_i = \eta \circ \alpha$$

sends R to the inner deterministic multirelation isomorphic to $\alpha(R)$. This is expressed in the following commuting diagram, obtained by replacing the dotted arrows in previous ones:



Set-theoretically, the determinisation maps are

$$\delta_o(R) = \{(a, B) \mid B = \bigcup R(a)\} \qquad \text{and} \qquad \delta_i(R) = \{(a, \{b\}) \mid b \in \bigcup R(a)\}.$$

Composing the bijections in the diagram above from bottom to top and vice versa yields the following corollary to Proposition 3.6.

Corollary 4.1. *The functors* δ_i : **DRel** $\to$ **IDRel** *and* δ_o : **IDRel** $\to$ **DRel** *are isomorphisms. They preserve the quantaloid structure with* $\bigcup$ *for inner deterministic multirelations and* $\biguplus$ *for outer deterministic ones.*

For outer deterministic multirelations, therefore, $\delta_o \circ \delta_i = id_{X \nrightarrow \mathcal{P}Y}$ and for inner deterministic ones, $\delta_i \circ \delta_o = id_{X \nrightarrow \mathcal{P}Y}$ and we get the universal property $R = \delta_i(S) \Leftrightarrow S = \delta_o(R)$ for inner deterministic R and outer deterministic S. By functoriality, $\delta_i(R * S) = \delta_i(R) * \delta_i(S)$ if R, S are outer deterministic and $\delta_o(R * S) = \delta_o(R) * \delta_o(S)$ if R, S are inner deterministic.

The determinisation maps also allow us to represent inner and outer deterministic multirelations as fixpoints.

Corollary 4.2. *The inner and outer deterministic multirelations are precisely the fixpoints of* δ_i *and* δ_o*, respectively.*

Proof. If R is inner deterministic, then $\delta_i(R) = \eta(\alpha(R)) = R$ by Lemma 3.4. If $\delta_i(R) = R$, then R is inner deterministic by Lemma 3.3. The proof for outer determinism is similar. □

4.2. Galois connections between relations and multirelations

The universal properties for α and Λ or η for relations and outer or inner deterministic multirelations generalise to Galois connections on arbitrary multirelations. These use $\subseteq$ on relations and $\sqsubseteq_\downarrow$ on multirelations. Beyond the identification of categories and quantaloids of deterministic multirelations, this is another main contribution of this article.

Proposition 4.3. *Let $R, S : X \nrightarrow \mathcal{P}Y$ and $T : X \nrightarrow Y$. Then*

1. $\alpha(R) \subseteq T \Leftrightarrow R \sqsubseteq_\downarrow \Lambda(T)$, $\eta(T) \sqsubseteq_\downarrow S \Leftrightarrow T \subseteq \alpha(S)$ *and* $\delta_i(R) \sqsubseteq_\downarrow S \Leftrightarrow R \sqsubseteq_\downarrow \delta_o(S)$,
2. $\alpha(R \Cap S) = \alpha(R) \cap \alpha(S)$,
3. δ_o *is a closure and* δ_i *an interior operator,*
4. (α, Λ) *and* (α, η) *are epi-mono-factorisations of* δ_o *and* δ_i*, both unique up to isomorphism,*
5. $\delta_o(R)$ *is the* $\sqsubseteq_\downarrow$*-least outer deterministic multirelation above* R *and* $\delta_i(R)$ *the* $\sqsubseteq_\downarrow$*-greatest inner deterministic multirelation below* R*.*

Proof. For (1), recall that $R \sqsubseteq_\downarrow S \Leftrightarrow R \subseteq S\Omega^\smile$ and $\Omega^\smile = \ni/\ni$. For the first Galois connection, $\alpha(R) \subseteq T \Leftrightarrow R \subseteq T/\ni$ using the standard Galois connection for left residuals. The claim then follows from $T/\ni = (\Lambda(T)\ni)/\ni = \Lambda(T)\Omega^\smile$, using a general law of residuals ($(RS)/Q = R(S/Q)$ for all composable relations R and S such that R is deterministic) in the last step. For the second Galois connection, first suppose $\eta(T) \subseteq S\Omega^\smile$. Then $T \subseteq S\Omega^\smile\ni = \alpha(S)$, using $\alpha \circ \eta = id$ and $\subseteq$-preservation of α in the first step, and $(R/R)R = R$, which holds for all relations R, in the second one. Conversely, suppose $T \subseteq \alpha(S)$. Then $\eta(T) \subseteq S\ni 1 \subseteq S\Omega^\smile$, because $\ni 1 \ni = \ni\alpha(\Lambda(Id)) = \ni$ and therefore $\ni 1 \subseteq \Omega^\smile$ by the Galois connection for left residuals. The third Galois connection is then immediate.

Item (2) follows from a simple set-theoretic calculation.

Property (3) would be standard for Galois connections over partial orders, but idempotency does not follow in general for preorders. In this specific instance, $\delta_o \circ \delta_o = \delta_o$ and $\delta_i \circ \delta_i = \delta_i$ hold because $\alpha \circ \Lambda = id = \alpha \circ \eta$. However, we give a more general proof for Galois connections where one of the two preorders is a partial order. Assume $f(x) \leq_B y \Leftrightarrow x \leq_A g(y)$ for $f : A \to B$ and $g : B \to A$ and preorders $\leq_A$ on A and $\leq_B$ on B. First, $f \circ g \circ f = f$ if $\leq_B$ is antisymmetric: $f(g(f(x))) \leq f(x)$ since $f \circ g$ is decreasing, and $f(x) \leq f(g(f(x)))$ since f is order-preserving and $g \circ f$ is increasing. Hence $f \circ g \circ f \circ g = f \circ g$ and $g \circ f \circ g \circ f = g \circ f$. By duality, $g \circ f \circ g = g$ if $\leq_A$ is antisymmetric, so $f \circ g$ and $g \circ f$ are idempotent also in this case.

Idempotency of δ_o follows from (1) by instantiating $f = \alpha$ and $g = \Lambda$ using $\subseteq$ as $\leq_B$. Idempotency of δ_i follows by instantiating $f = \eta$ and $g = \alpha$ using $\subseteq$ as $\leq_A$. The remaining requirements of Galois connections follow from (1) also for preorders.

For (4), surjectivity of α and injectivity of Λ and η is immediate from $\alpha \circ \Lambda = id = \alpha \circ \eta$. For uniqueness, note that every function in **Set** has this property, and the proof is standard.

For (5), $R \sqsubseteq_\downarrow \delta_o(R)$ by (3). Now suppose $\delta_o(S) = S$ and $R \sqsubseteq_\downarrow S$. Then $\delta_o(R) \sqsubseteq_\downarrow \delta_o(S) = S$ by order-preservation of δ_o. The proof for δ_i is similar. □

Property (1) can be dualised using inner complementation to obtain Galois connections for $\sqsubseteq_\uparrow$; see also Section 7. The properties in (1) and (2) can be summarised in the language of topos theory by saying that the adjunction (α, Λ) is an essential geometric morphism – but of course **Rel** does not form a topos. Property (3) shows that δ_o is a monad and δ_i a comonad on $\sqsubseteq_\downarrow$, both of which are idempotent. As usual, the unit and counit are arrows $R \sqsubseteq_\downarrow \delta_o(R)$ and $\delta_i(R) \sqsubseteq_\downarrow R$, the multiplication and comultiplication are arrows $\delta_o(\delta_o(R)) \sqsubseteq_\downarrow \delta_o(R)$ and $\delta_i(R) \sqsubseteq_\downarrow \delta_i(\delta_i(R))$.

4.3. Expressing the determinisation maps in a multirelational language

With a view on a multirelational language, it is worth noting that δ_o and δ_i can be expressed in multirelational terms without going back to general relations or power allegories. This requires two additional concepts from the inner structure: the set of *co-atoms* $\mathsf{A}_\Cap = {\sim}\mathsf{A}_\Cup = \{(a, Y - \{b\}) \mid a \in X \wedge b \in Y\}$ in $X \nrightarrow \mathcal{P}Y$ and the *duality* operation $R^{\mathsf{d}} = -{\sim}R$, which relates the inner and the outer structure [5].

Lemma 4.4. *Let $R : X \nrightarrow \mathcal{P}Y$. Then*

1. $\delta_i(R) = R{\downarrow} \cap \mathsf{A}_{\Cup}$,
2. $\delta_o(R){\downarrow} = -((-\delta_i(R) \cap \mathsf{A}_{\Cup}){\uparrow}) = -((-(R{\downarrow}) \cap \mathsf{A}_{\Cup}){\uparrow})$,
3. $\delta_o(R){\uparrow} = \delta_i(R){\uparrow}^{\mathsf{d}} = -((\sim\delta_i(R)){\downarrow}) = -((\sim(R{\downarrow}) \cap \mathsf{A}_{\Cap}){\downarrow})$,
4. $\delta_o(R) = -((-(R{\downarrow}) \cap \mathsf{A}_{\Cup}){\uparrow}) \cap -((\sim(R{\downarrow}) \cap \mathsf{A}_{\Cap}){\downarrow})$.

Proof. For (1), $R{\downarrow} \cap \mathsf{A}_{\Cup} = R\Omega^{\smile} \cap U1 = R(\Omega^{\smile} \cap U1)$, hence it suffices to show

$$\ni 1 = \Omega^{\smile} \cap U1 = (\ni/\ni) \cap U1.$$

The inclusion $\subseteq$ follows by residuation from $\ni 1 \ni = \alpha(\eta(\ni)) = \ni$. The opposite inclusion follows from $-\ni 1 \subseteq -\ni\in = -(\ni/\ni)$ using boolean properties.

For (2),

$$\begin{aligned}
-((-\delta_i(R) \cap \mathsf{A}_{\Cup}){\uparrow}) &= -(-(R\ni 1)1^{\smile}1\Omega) \\
&= -(-(R\ni 11^{\smile})\in) \\
&= -(-(R\ni)\in) \\
&= R\ni/\ni \\
&= \Lambda(R\ni)\ni/\ni \\
&= \Lambda(R\ni)(\ni/\ni) \\
&= \delta_o(R){\downarrow}
\end{aligned}$$

using Lemma 2.9 in the first step. The second equality follows by (1) and boolean algebra.

For (3),

$$\begin{aligned}
\delta_i(R){\uparrow}^{\mathsf{d}} &= -\sim(\delta_i(R){\uparrow}) \\
&= -((\sim\delta_i(R)){\downarrow}) \\
&= -(\delta_i(R)C\Omega^{\smile}) \\
&= -(\alpha(R)1\Omega C) \\
&= -(\alpha(R)\in C) \\
&= -(\alpha(R)(-\in)) \\
&= \alpha(R)^{\smile}\backslash\in \\
&= \Lambda(\alpha(R))\Omega \\
&= \delta_o(R){\uparrow}
\end{aligned}$$

using Lemma 2.3. The remaining equality follows again by (1).

Item (4) follows from (2) and (3) since $\delta_o(R)$ is convex-closed. □

4.4. Properties of approximation and determinisation

In light of Lemma 3.5 and Corollary 4.1 it is natural to ask how α translates arbitrary Peleg compositions of multirelations and likewise for δ_o and δ_i, which are based on α. It can be expected that these are no longer strict functors. Here we show that they are at least (op)lax.

First we show how item (3) in the proof of Lemma 3.5 generalises to arbitrary multirelations.

Lemma 4.5. *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : Y \nrightarrow \mathcal{P}Z$. Then*

1. $\alpha(R_*) = \alpha(dom(R)_*)\alpha(R)$,
2. $\alpha(R * S) \subseteq \alpha(R)\alpha(S)$,
3. $\alpha(R{\downarrow}) = \alpha(R)$.

Proof. For (1), $\ni\alpha(T) = \alpha(\Lambda(\ni T\ni)) = \alpha(T_{\mathcal{P}})$. Thus

$$\begin{aligned}\alpha(R_*) &= dom(R)_* \bigcup_{T\subseteq_d R} \alpha(T_{\mathcal{P}}) \\ &= dom(R)_* \bigcup_{T\subseteq_d R} \ni\alpha(T) \\ &= dom(R)_* \ni\alpha(R) \\ &= \alpha(dom(R)_*)\alpha(R).\end{aligned}$$

This implies (2) because $\alpha(R * S) = R\alpha(S_*) \subseteq R\alpha(1)\alpha(S) = \alpha(R)\alpha(S)$.

For (3), $\alpha(R\downarrow) = R\Omega^{\smile}\ni = R\ni = \alpha(R)$ since $\in\Omega = \in$. □

Part (2) above cannot be strengthened to an equality. Structure is lost by approximation.

Example 4.6. For $R = \{(a, \{a, b\})\}$,

$$\alpha(R * R) = \alpha(\emptyset) = \emptyset \subset \{(a, a), (a, b)\} = \alpha(R) = \alpha(R)\alpha(R).$$

Lemma 4.5 and Example 4.6 are important for the study of modal operators on multirelations in the third part of this trilogy [4]. Before proving an analogous lemma for δ_o and δ_i we note another property of these maps that is needed in [4].

Lemma 4.7. *$\delta_i \circ \delta_i = \delta_i$, $\delta_o \circ \delta_o = \delta_o$, $\delta_i \circ \delta_o = \delta_i$ and $\delta_o \circ \delta_i = \delta_o$.*

Proof. The first two properties are part of the closure conditions in Proposition 4.3. The proof of the remaining ones are similar. □

Lemma 4.8. *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : Y \to \mathcal{P}Z$. Then*

1. $\delta_i(R) * S = \alpha(R)S$,
2. $\delta_i(R * S) \subseteq \delta_i(R) * \delta_i(S)$,
3. $\delta_o(R * S) \sqsubseteq_{\updownarrow} \delta_o(R) * \delta_o(S)$.

Proof. For (1), $\delta_i(R) * S = \alpha(R)1S_* = \alpha(R)(1 * S) = \alpha(R)S$.

For (2), $\delta_i(R * S) = \eta(\alpha(R * S)) \subseteq \eta(\alpha(R)\alpha(S)) = \eta(\alpha(R)) * \eta(\alpha(S)) = \delta_i(R) * \delta_i(S)$ using Lemmas 4.5 and 3.5.

For (3), $\delta_o(R * S) = \Lambda(\alpha(R * S)) \sqsubseteq_{\updownarrow} \Lambda(\alpha(R)\alpha(S)) = \Lambda(\alpha(R)) * \Lambda(\alpha(S)) = \delta_o(R) * \delta_o(S)$ using Lemmas 4.5, 3.8 and 3.5. □

Item (3) highlights once again the relevance of the Egli-Milner preorder.

The final lemma of this section relates outer determinism with the Peleg lifting and other standard notions of power allegories. Its proof is immediate from properties of Section 2.

Lemma 4.9. *Let $R : X \nrightarrow \mathcal{P}Y$. Then*

1. $R_{\mathcal{P}} = \delta_o(\ni R)$,
2. $\delta_o(R) = \eta R_{\mathcal{P}} = \Lambda(R)\mu$.

5. Category of Inner Univalent Multirelations

It remains to describe the category of inner univalent multirelations. As it is not isomorphic to **Rel**, we include different multirelational techniques.

Inner total multirelations have previously been called *non-terminal* [8], writing $\nu(R)$ for the set of *non-terminal* elements of R: those pairs in R whose second component is not $\emptyset$, that is, $\nu(R) = R - 1_{\Cup}$. In addition, the map $\tau(R) = R * \emptyset = R \cap 1_{\Cup}$ projects on the *terminal* elements of R: those pairs in R whose second component is $\emptyset$. Every multirelation can thus be decomposed into its terminal and non-terminal part:

$$R = \nu(R) \cup \tau(R) \qquad \text{and} \qquad \nu(R) \cap \tau(R) = \emptyset.$$

Note that products and coproducts in **Rel** coincide – both are disjoint unions – so that we may write $R = (\nu(R), \tau(R))$.

Lemma 5.1 ([8])**.** *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : Y \nrightarrow \mathcal{P}Z$. Then*

1. $R * S = \nu(R) * S \cup \tau(R)$,
2. $\tau(R * S) = \tau(R) \cup \nu(R) * \tau(S)$,
3. $\tau(\nu(R)) = \emptyset = \nu(\tau(R))$.

For inner univalent multirelations, the above decomposition properties simplify. Intuitively, the non-terminal part of an inner univalent multirelation is inner deterministic: those elements that are not related to the empty set must be related to a singleton set. Inner univalent multirelations can thus be decomposed into an inner deterministic and a terminal part. Proving this fact in our relational language requires a few simple properties.

Lemma 5.2. *Let $R : X \nrightarrow \mathcal{P}Y$. Then*

1. $\alpha(\tau(R)) = \emptyset$ *and* $\alpha(\nu(R)) = \alpha(R)$,
2. $\nu(\delta_i(R)) = \delta_i(R) = \delta_i(\nu(R))$ *and* $\tau(\delta_i(R)) = \emptyset$.

Proof. For (1), $\alpha(\tau(R)) = R\alpha(\emptyset_*) = R\alpha(\emptyset_*)\alpha(\emptyset) = R\alpha(\emptyset_*)\emptyset = \emptyset$ using Lemma 4.5(1). The second property follows from $R = \nu(R) \cup \tau(R)$. Both properties in (2) are obvious. □

Lemma 5.3. *Let $R : X \nrightarrow \mathcal{P}Y$. Then the following statements are equivalent:*

1. *R is inner univalent,*
2. *$\nu(R)$ is inner deterministic,*
3. $\nu(R) = \delta_i(R)$.

Proof. If R is inner univalent, that is, $R \subseteq 1_{\Cup} \cup \mathsf{A}_{\Cup}$, then $\nu(R) = R - 1_{\Cup} \subseteq \mathsf{A}_{\Cup}$, thus $\nu(R) = \nu(R) \cap \mathsf{A}_{\Cup}$ and (2) holds. In this case, $\nu(R)$ is a fixpoint of δ_i by Corollary 4.2 and (3) follows immediately by Lemma 5.2(2). Thus $R - 1_{\Cup} = R{\downarrow} \cap \mathsf{A}_{\Cup}$ by Lemma 4.4 and therefore $R \subseteq \mathsf{A}_{\Cup} \cup 1_{\Cup}$, which implies (1). □

We can therefore rewrite $R = (\delta_i(R), \tau(R))$ as expected. Based on this we now show that the Peleg composition of inner univalent multirelations can be represented as a generalisation of the standard semidirect product of two monoids, where the first monoid is replaced by the category **IDRel** and the second by the partial monoid of the terminal multirelations under unions and with the empty multirelations as units. The units of this generalised semidirect product are the pairs $(1, \emptyset)$ and it remains to consider the multiplication.

A stepping stone is the following lemma, which generalises parts of Lemma 3.5 and Corollary 4.1 from inner determinism to inner univalence, and is thus of independent interest.

Lemma 5.4. *Let $R : X \nrightarrow \mathcal{P}Y$ be inner univalent and $S : Y \nrightarrow \mathcal{P}Z$. Then $\alpha(R * S) = \alpha(R)\alpha(S)$ and $\delta_i(R * S) = \delta_i(R) * \delta_i(S)$.*

Proof. Lemmas 5.1, 5.3(3) and 4.8(1) imply that $R * S = \alpha(R)S \cup \tau(R)$ if R is inner univalent. So $\alpha(R * S) = \alpha(\alpha(R)S \cup \tau(R)) = \alpha(\alpha(R)S) \cup \alpha(\tau(R)) = \alpha(R)\alpha(S)$ by Lemma 5.2(1). The second identity is then immediate from Lemma 3.5. □

Proposition 5.5. *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : Y \nrightarrow \mathcal{P}Z$ be inner univalent. Then*

$$R * S = (\delta_i(R) * \delta_i(S), \delta_i(R) * \tau(S) \cup \tau(R))\,.$$

Proof. To establish the universal property of the coproduct it suffices to check that

$$\delta_i(R) * \delta_i(S) = \nu(R * S) \qquad \text{and} \qquad \delta_i(R) * \tau(S) \cup \tau(R) = \tau(R * S).$$

First, $R * S = \alpha(R)S \cup \tau(R) = \alpha(R)\delta_i(S) \cup \alpha(R)\tau(S) \cup \tau(R) = \delta_i(R) * \delta_i(S) \cup \tau(R * S)$ using Lemmas 5.1(1), 5.3(3), 4.8(1) and 5.1(2). Thus, $\nu(R * S) = \nu(\delta_i(R) * \delta_i(S)) = \delta_i(R) * \delta_i(S)$, using the definition of ν and Lemmas 5.1(3), 5.4 and 5.2(2). The second property is immediate from Lemmas 5.1(2) and 5.3(3). □

The product in Proposition 5.5 has precisely the shape of the multiplication of a semidirect product. The main result of this section is now straightforward.

Proposition 5.6. *The inner univalent multirelations form a category with respect to Peleg composition and the 1_X.*

Proof. The universal property of the coproduct in Proposition 5.5 and Lemma 5.4 guarantee that Peleg composition preserves inner univalence ($\nu(R * S) = \delta_i(R * S)$ whenever R, S are inner univalent); the 1_X are clearly inner univalent. Associativity of Peleg composition (for composable inner univalent multirelations) is routine: its proof is that of the standard semidirect product construction of monoids, plus some type checking. □

We call this category **IURel**. We have thus characterised the categories of inner and outer deterministic multirelations, **IDRel** and **DRel**, as well as those of inner and outer univalent multirelations **IURel** and **URel**, within our multirelational language. While the results for outer deterministic and univalent multirelations were known, our proofs present new structural insights. The results for their inner counterparts are new.

Remark 5.7. For an explicit proof of associativity of the τ-component of Peleg composition, suppose R, S and T are inner univalent. Then, using Lemma 5.4 and techniques from the proof of Proposition 5.5,

$$\begin{aligned}\tau(R * (S * T)) &= \tau(R) \cup \alpha(R)\tau(S) \cup \alpha(R)\alpha(S)\tau(T)\\ &= \tau(R) \cup \alpha(R)\tau(S) \cup \alpha(R * S)\tau(T)\\ &= \tau((R * S) * T).\end{aligned}$$

The proof for the ν-component is simply associativity of Peleg composition in **IDRel**.

Remark 5.8. The proof of $\nu(R*S) = \delta_i(R)*\delta_i(S)$ in Proposition 5.5 can be adapted to show that $\nu(R*S) = \nu(R)*\nu(S)$ if R is inner univalent: $\nu(R*S) = \nu(\alpha(R)S) = \alpha(R)\nu(S) = \nu(R)*\nu(S)$. The second equality holds since $\nu(QS) = Q\nu(S)$ for arbitrary Q, S by standard relational properties.

Lemma 5.9. *Each homset in* **IURel** *forms a complete lattice. In this category, Peleg composition preserves arbitrary sups in its first argument and non-empty sups in the second.*

Proof. Inner univalent multirelations are closed under arbitrary unions by Lemma 5.3, as ν and δ_i preserve arbitrary unions. This yields a complete lattice structure on homsets. As already mentioned, Peleg composition preserves arbitrary unions in its first argument. Preservation in the second argument, in the inner univalent case, holds for non-empty unions: if $I \neq \emptyset$, then

$$R * \bigcup_{i\in I} S_i = \left(\alpha(R) \bigcup_{i\in I} S_i\right) \cup \tau(R) = \bigcup_{i\in I} (\alpha(R)S_i \cup \tau(R)) = \bigcup_{i\in I} R * S_i.$$

□

Example 5.10. Inner univalent multirelations do not form a quantaloid with respect to sups, that is, Proposition 3.10 does not generalise beyond Lemma 5.9. For $I = \emptyset$ we have $R * \bigcup_{i\in I} S_i = R * \emptyset$ and $\bigcup_{i\in I}(R * S_i) = \emptyset$. But $R * \emptyset = \emptyset$ if and only if $\tau(R) = \emptyset$, that is, R must be inner total. This shows that R must be inner deterministic for this argument to work.

Likewise, outer univalent multirelations do not form quantaloids with respect to $\Cup$. A counterexample again uses $I = \emptyset$ in which case $\emptyset * \Cup_{i\in I} S_i = \emptyset$ but $\Cup_{i\in I}(\emptyset * S_i) = 1_{\mathbb{U}}$.

Remark 5.11. In light of Lemmas 5.2 and 5.4 one might wonder whether similar properties hold for δ_o. Indeed, $\delta_o(\nu(R)) = \delta_o(R)$ for all R and $\delta_o(R * S) = \delta_o(R) * \delta_o(S)$ for inner univalent R hold, but $\nu(\delta_o(\emptyset)) = \nu(1_{\mathbb{U}}) = \emptyset \neq 1_{\mathbb{U}} = \delta_o(\emptyset)$ because δ_o adds a pair $(a, \emptyset)$ for each a that is not related to any set.

Example 5.12. Lemma 5.4, Proposition 5.6 and the previous remark show that α, δ_i and δ_o are functors of type **IURel** $\rightarrow$ **Rel**, **IURel** $\rightarrow$ **IDRel** and **IURel** $\rightarrow$ **DRel**, respectively. They need not be injective. For $X = \{a, b\}$, for instance, α maps the inner univalent multirelations $\{(a, \emptyset)\}$ and $\{(a, \emptyset), (b, \emptyset)\}$ in $X \rightarrow \mathcal{P}Y$ to the relation $\emptyset_{X,Y}$. This failure of injectivity extends along Λ and η, so that the categories are not isomorphic.

6. A Fine-Grained View on Determinisation

Many properties of inner deterministic multirelations hold already for inner univalent ones. Here we prove refined results. First we refine Corollary 4.2 for outer deterministic multirelations.

Lemma 6.1.

1. *The outer univalent multirelations are precisely the postfixpoints of δ_o with respect to $\subseteq$ and the prefixpoints of δ_o with respect to $\sqsubseteq_\uparrow$.*
2. *Prefixpoints of δ_o with respect to $\subseteq$ and $\sqsubseteq_\downarrow$ and postfixpoints with respect to $\sqsubseteq_\uparrow$ are outer total. The postfixpoints of δ_o with respect to $\sqsubseteq_\updownarrow$ and $\sqsubseteq_\uparrow$ coincide.*
3. *Prefixpoints of δ_o with respect to $\sqsubseteq_\updownarrow$ are outer deterministic.*

Proof. For (1), if R is outer univalent, then

$$R = R(\in \div \in) = (\in R^\smile \div \in) \cap RU \subseteq \in R^\smile \div \in = \Lambda(\alpha(R)) = \delta_o(R).$$

Conversely, if R is a postfixpoint of δ_o with respect to $\subseteq$, then

$$R^\smile R \subseteq \delta_o(R)^\smile \delta_o(R) = \Lambda(\alpha(R))^\smile \Lambda(\alpha(R)) = (\in \div \in R^\smile)(\in R^\smile \div \in) \subseteq \in \div \in = Id.$$

Since $\delta_o(R) \subseteq \delta_o(R)\uparrow$, postfixpoints of δ_o with respect to $\subseteq$ are also prefixpoints with respect to $\sqsubseteq_\uparrow$. Conversely, if R is a prefixpoint with respect to $\sqsubseteq_\uparrow$, then

$$R \subseteq \delta_o(R)\uparrow = \Lambda(\alpha(R))\Omega = \Lambda(\alpha(R))(\in \backslash \in) = \in \Lambda(\alpha(R))^\smile \backslash \in = \in(\in \div \in R^\smile) \backslash \in = \in R^\smile \backslash \in.$$

Together with $R \subseteq \alpha(R)/\ni$, we obtain $R \subseteq \in R^\smile \div \in = \Lambda(\alpha(R)) = \delta_o(R)$. Thus R is a postfixpoint with respect to $\subseteq$.

For (2), if R is a postfixpoint of δ_o with respect to $\sqsubseteq_\uparrow$, then $\delta_o(R) \subseteq R\uparrow$. Hence outer totality of R follows by $U = \Lambda(\alpha(R))U = \delta_o(R)U \subseteq R\uparrow U = R\Omega U = RU$ using that Λ yields outer deterministic multirelations and Ω is outer total. The proof for prefixpoints with respect to $\sqsubseteq_\downarrow$ is similar, using $\Omega^\smile$ instead of Ω. Moreover prefixpoints with respect to $\subseteq$ are also postfixpoints with respect to $\sqsubseteq_\uparrow$ since $R \subseteq R\uparrow$. The remaining claim follows since any R is a postfixpoint of δ_o with respect to $\sqsubseteq_\downarrow$ (Proposition 4.3).

Finally, (3) follows by (1) and (2). □

Obviously, if R is outer deterministic, then $\delta_o(R) = \Lambda(\alpha(R)) = R$. The converse implication follows by (3) above. This yields an alternative algebraic proof of the fact that the outer deterministic multirelations are precisely the fixpoints of δ_o.

Next we refine Corollary 4.2 for inner deterministic multirelations.

Lemma 6.2.

1. *Inner univalent multirelations are prefixpoints of δ_i with respect to $\subseteq$ and postfixpoints of δ_i with respect to $\sqsubseteq_\uparrow$.*
2. *Postfixpoints of δ_i with respect to $\sqsubseteq_\downarrow$ are inner univalent. The postfixpoints of δ_i with respect to $\sqsubseteq_\updownarrow$ and $\sqsubseteq_\downarrow$ coincide.*
3. *The inner total multirelations are precisely the prefixpoints of δ_i with respect to $\sqsubseteq_\uparrow$. The prefixpoints of δ_i with respect to $\sqsubseteq_\updownarrow$ and $\sqsubseteq_\uparrow$ coincide.*
4. *Postfixpoints of δ_i with respect to $\subseteq$ are inner deterministic.*

Proof. (1) follows by Lemma 5.3, using $\delta_i(R) \subseteq R \subseteq R\uparrow$.

For (2), if R is a postfixpoint of δ_i with respect to $\sqsubseteq_\downarrow$, then $R \subseteq \delta_i(R)\downarrow \subseteq \mathsf{A}_{\mathbb{U}}\downarrow = 1_{\mathbb{U}} \cup \mathsf{A}_{\mathbb{U}}$, so R is inner univalent. Hence by (1), R is also a postfixpoint with respect to $\sqsubseteq_\uparrow$ and therefore with respect to $\sqsubseteq_\updownarrow$.

For (3), if R is a prefixpoint of δ_i with respect to $\sqsubseteq_\uparrow$, then $R \subseteq \delta_i(R)\uparrow \subseteq \mathsf{A}_{\mathbb{U}}\uparrow = -1_{\mathbb{U}}$, so R is inner total. Conversely, if R is inner total, then

$$R \subseteq RR^\smile R \subseteq R(-1_{\mathbb{U}})^\smile(-1_{\mathbb{U}}) = R{\ni}{\in} = R{\ni}1\Omega = \delta_i(R)\Omega = \delta_i(R)\uparrow.$$

The claim follows since any R is a prefixpoint of δ_i with respect to $\sqsubseteq_\downarrow$ by Proposition 4.3.

For (4), since $\delta_i(R) \subseteq \delta_i(R)\downarrow$ and $\delta_i(R) \subseteq \delta_i(R)\uparrow$, postfixpoints of δ_i with respect to $\subseteq$ are also postfixpoints with respect to $\sqsubseteq_\downarrow$ and prefixpoints with respect to $\sqsubseteq_\uparrow$. Hence the final claim follows by (2) and (3). □

To show that the inner deterministic multirelations are precisely the fixpoints of δ_i, it remains to check, using parts (1) and (4) above, that inner deterministic multirelations are postfixpoints of δ_i with respect to $\subseteq$. Indeed, if R is inner deterministic, then $R \subseteq \mathsf{A}_{\mathbb{U}} = U1$. Hence

$$R = R \cap U1 \subseteq R1^\smile 1 \subseteq R{\ni}1 = \delta_i(R)$$

using $1 \subseteq {\in}$.

The following results revisit previous closure results in the context of total multirelations.

Lemma 6.3. *Inner and outer total multirelations are closed under Peleg composition.*

Proof. Let R and S be outer total. Then

$$S_* = dom(S)_* \bigcup_{Q \subseteq_d S} Q_{\mathcal{P}} = 1_* \bigcup_{Q \subseteq_d S} Q_{\mathcal{P}} = \bigcup_{Q \subseteq_d S} ({\in}Q{\in} \div {\in}).$$

Hence

$$S_* S_*^\smile = \bigcup_{P,Q \subseteq_d S} ({\in}P{\in} \div {\in})({\in} \div {\in}Q{\in}) = \bigcup_{P,Q \subseteq_d S} ({\in}P{\in} \div {\in}Q{\in}) \supseteq \bigcup_{P \subseteq_d S} ({\in}P{\in} \div {\in}P{\in}) \supseteq Id.$$

Thus $(R * S)(R * S)^\smile = RS_*S_*^\smile R^\smile \supseteq RR^\smile \supseteq Id$.

Let R and S be inner total, that is, $R \subseteq -1_{\mathbb{U}}$ and $S \subseteq -1_{\mathbb{U}}$. Since Peleg composition preserves $\subseteq$ it suffices to show $-1_{\mathbb{U}} * -1_{\mathbb{U}} \subseteq -1_{\mathbb{U}}$. We have

$$-1_{\mathbb{U}} * -1_{\mathbb{U}} = -1_{\mathbb{U}}(-1_{\mathbb{U}})_* = -1_{\mathbb{U}} dom(-1_{\mathbb{U}})_* \bigcup_{Q \subseteq_d -1_{\mathbb{U}}} Q_{\mathcal{P}} = -1_{\mathbb{U}} \bigcup_{Q \subseteq_d -1_{\mathbb{U}}} Q_{\mathcal{P}} = \bigcup_{Q \subseteq_d -1_{\mathbb{U}}} -1_{\mathbb{U}} Q_{\mathcal{P}}.$$

Hence it remains to show $-1_{\mathbb{U}} Q_{\mathcal{P}} \subseteq -1_{\mathbb{U}}$ for any $Q \subseteq_d -1_{\mathbb{U}}$. The latter condition means Q is outer univalent, outer total and inner total. The remaining goal is equivalent to $1_{\mathbb{U}}(Q_{\mathcal{P}})^\smile \subseteq 1_{\mathbb{U}}$. This follows by

$$\begin{aligned}
1_{\mathbb{U}}(Q_{\mathcal{P}})^\smile &= 1_{\mathbb{U}} \Lambda({\ni}Q{\ni})^\smile \\
&= 1_{\mathbb{U}}({\in} \div {\in}Q^\smile{\in}) \\
&= ({\in}1_{\mathbb{U}}{}^\smile \div {\in}Q^\smile{\in}) \\
&= (\emptyset \div {\in}Q^\smile{\in}) \\
&= \emptyset / {\ni}Q{\ni} \\
&= -(U{\in}Q^\smile{\in}) \\
&= -(U(-1_{\mathbb{U}}Q^\smile{\in})) \\
&\subseteq -(UQQ^\smile{\in}) \\
&\subseteq -(U{\in}) \\
&= 1_{\mathbb{U}}
\end{aligned}$$

using that Q is inner and outer total. □

Hence closure of inner deterministic multirelations under Peleg composition (Proposition 3.6) also follows by combining Proposition 5.6 and Lemma 6.3. However, composition of outer total multirelations or inner total multirelations, respectively, need not be associative.

Example 6.4. Let $X = \{a, b\}$ and $R, S : X \nrightarrow \mathcal{P}X$ with $R = \{(a, \{a, b\}), (b, \{a, b\})\}$ and $S = \{(a, \emptyset), (b, \{a\}), (b, \{b\})\}$. Then $(a, \{a, b\}) \in R * (R * S) - (R * R) * S$. The same holds for $R = \{(a, \{a\}), (a, \{a, b\}), (b, \{a\})\}$ and $S = \{(a, \{a\}), (a, \{b\})\}$. Thus neither outer nor inner total multirelations form categories.

We conclude with preservation properties for outer total multirelations.

Lemma 6.5. *Let $R : X \nrightarrow \mathcal{P}Y$ and $S : Y \nrightarrow \mathcal{P}Z$ be outer total. Then*

1. $\alpha(R * S) = \alpha(R)\alpha(S)$,
2. $\delta_i(R * S) = \delta_i(R) * \delta_i(S)$ *and* $\delta_o(R * S) = \delta_o(R) * \delta_o(S)$.

Proof. For (1), if S is outer total, then $dom(S) = Id$ and $\alpha(S_*) = \alpha(Id)\alpha(S)$ by Lemma 4.5(1). The inclusion step in the proof of Lemma 4.5(2) then becomes an equality, which shows the claim. Item (2) is then immediate from (1) and Lemma 3.5. □

Example 6.6. Part (1) of Lemma 6.5 does not hold for outer univalent multirelations: we have $\alpha(1 * \emptyset) = \alpha(1) = Id$ but $\alpha(1)\alpha(\emptyset) = Id\,\emptyset = \emptyset$. For (2) consider $X = \{a, b\}$ and $R, S : X \nrightarrow \mathcal{P}X$ with $R = \{(a, \{a, b\})\}$ and $S = \{(a, \{a\})\}$. Then $\delta_i(R * S) = \delta_i(\emptyset) = \emptyset$ but

$$\delta_i(R) * \delta_i(S) = \{(a, \{a\}), (a, \{b\})\} * S = \{(a, \{a\})\}.$$

Moreover $\delta_o(R * S) = \delta_o(\emptyset) = \{(a, \emptyset), (b, \emptyset)\}$ but

$$\delta_o(R) * \delta_o(S) = (R \cup \{(b, \emptyset)\}) * (S \cup \{(b, \emptyset)\}) = \{(a, \{a\}), (b, \emptyset)\}.$$

7. Co-Determinisation and Further Isomorphisms

We discuss two additional operations called *co-fusion* and *co-fission* of multirelations. They are obtained via the inner isomorphism:

$$\tilde{\delta_o}(R) = {\sim}\delta_o({\sim}R) \qquad \text{and} \qquad \tilde{\delta_i}(R) = {\sim}\delta_i({\sim}R).$$

It follows that

$$\tilde{\delta_o}(R) = \left\{(a, B) \mid B = \bigcap R(a)\right\} \qquad \text{and} \qquad \tilde{\delta_i}(R) = R{\uparrow} \cap \mathsf{A}_{\Cap}.$$

Other properties of co-fusion and co-fission follow immediately from the inner isomorphism. For instance,

$$\tilde{\delta_o}(R) = -(-R{\uparrow} \cap \mathsf{A}_{\Cap}){\downarrow} \cap -({\sim}R{\uparrow} \cap \mathsf{A}_{\Cup}){\uparrow}$$

and similar to property (1) of Proposition 4.3 there is a Galois connection with respect to $\sqsubseteq_{\uparrow}$. See our Isabelle theories for details [10].

In Section 4 we have used isomorphisms to represent $\alpha(R)$ as outer and inner deterministic multirelations $\delta_o(R)$ and $\delta_i(R)$. We now discuss further isomorphic representations. Since outer deterministic multirelations are isomorphic to their down-closures, we obtain the down-closed representation $\delta_{\downarrow}(R) = \delta_o(R){\downarrow}$. Application of fusion takes us back according to $\delta_o(R) = \delta_o(\delta_{\downarrow}(R))$. Moreover, fission is now obtained by $\delta_i(R) = \delta_{\downarrow}(R) \cap \mathsf{A}_{\Cup}$. Hence the down-closed representation contains both the fusion and the fission. Finally, $(R * S){\downarrow} = R * S{\downarrow} = R * (1_{\Cup} \cup S{\downarrow}) = R{\downarrow} * S{\downarrow}$ for outer deterministic S and $(-){\downarrow}$ preserves arbitrary inner unions of outer deterministic multirelations. Hence we obtain an isomorphism between the quantaloid of outer deterministic multirelations with $\biguplus$ and the quantaloid of down-closures of outer deterministic multirelations with $\biguplus$.

Outer deterministic multirelations are also isomorphic to their up-closures, so we obtain the up-closed representation $\delta_{\uparrow}(R) = \delta_o(R)\uparrow$. To get back we apply co-fusion: $\delta_o(R) = \tilde{\delta_o}(\delta_{\uparrow}(R))$. Furthermore, co-fission is obtained by $\tilde{\delta_i}(R) = \delta_{\uparrow}(R) \cap \mathsf{A}_{\Cap}$. Hence the up-closed representation contains both the fusion and the co-fission. By [5, Example 4.9], up-closure does not distribute over Peleg composition for outer deterministic multirelations, so there is no quantaloid isomorphism in this case. Nevertheless, since $(-)\uparrow$ preserves arbitrary inner unions of outer deterministic multirelations, we obtain complete lattice isomorphisms between outer deterministic multirelations and their up-closures.

The range of $\tilde{\delta_o}(-)$ equals that of δ_o, namely the outer deterministic multirelations. Hence similar results are obtained by starting from $\tilde{\delta_o}(R)$ instead of $\delta_o(R)$ and considering down-/up-closed representations. From the range of δ_i it is also possible to apply $\downarrow$ or $\uparrow$ and then go back by δ_i or intersection with $\mathsf{A}_{\Cup}$. A similar construction applies to the range of $\tilde{\delta_i}(-)$. Note that $(R * S)\uparrow = R\uparrow * S\uparrow$ for inner deterministic R [5, Lemma 4.8] and $(-)\uparrow$ preserves arbitrary unions of inner deterministic multirelations [5, Lemma 4.3]. Hence we obtain an isomorphism between the quantaloid of inner deterministic multirelations with $\bigcup$ and the quantaloid of up-closures of inner deterministic multirelations with $\bigcup$. Example 4.10 in [5] rules out a corresponding result for the down-closure of inner deterministic multirelations. However, since $(-)\downarrow$ preserves arbitrary unions of inner deterministic multirelations, at least we obtain complete lattice isomorphisms between inner deterministic multirelations and their down-closures.

8. Conclusion

We have studied the inner structure of multirelations through the categories of outer and inner univalent and deterministic multirelations and determinisation maps in a multirelational language that combines features of relation algebra and power allegories with multirelational concepts. These show that the power transpose of power allegories together with the maps that postcompose with the converse membership relation and with the unit of the Peleg composition of multirelations, as well as the maps projecting on the terminal part of a multirelation play important structural roles in their study. Our results add to previous work on the various inner and outer operations on multirelations [5], but shift the focus towards univalence and determinism and from a relation-algebraic language to a language based on power allegories.

Another contribution of our work, only mentioned briefly in the introduction of this article, are our Isabelle components for multirelations [10]. These contain a comprehensive study of the basic laws for multirelations, including most of the concepts and results in this paper: from the inner operations, closures and preorders, the Kleisli and Peleg lifting and Peleg composition, to relational and multirelational properties of concrete power allegories (in **Rel**), the determinisation maps and their properties, and beyond. The PDF proof document in the Archive of Formal Proofs contains almost 200 pages. It thus constitutes at least a medium-size case study in formalised mathematics and can be used and extended by anyone interested in formal reasoning with multirelations.

Our multirelational language and its formalisation with Isabelle has so far been based on concrete relations and multirelations. An axiomatic extension of the abstract allegorical approach, which equips boolean power allegories with a basis of multirelational operations, is the most natural continuation of this work. This would extend Bird and de Moor's algebra of programming [2] to alternating nondeterminism. We expect that the results in our trilogy of articles and our Isabelle components prepare the ground for this work. Isabelle can be particularly helpful with rapidly prototyping axiom systems and checking the redundancy of axioms using deduction as well as their irredundancy using counterexample search. The characterisation of intuitionistic variants of power allegories based on locally complete allegories is another interesting question, or the consideration of categories of multirelations in arbitrary Grothendieck topoi.

Computations with only demonic nondeterminism have been modelled using binary relations [1]. These arise as a subclass of up-closed multirelations with Parikh composition [17]. Here, multirelations are not required to be up-closed and we use Peleg composition instead of Parikh composition, since the former has the right structural properties for our purposes. While outer deterministic multirelations represent purely demonic nondeterminism and inner union models demonic choice, Peleg composition differs from demonic composition.

In computer science, nondeterminism and partiality are often modelled using monads. An obvious alternative to the relational and allegorical approach in this work is therefore the translation of multirelations in **Rel** into doubly nondeterministic functions in **Set** and the application of the standard monadic machinery in **Set**. While this approach may be accessible to a wider range of readers, its structural complexity might be similar to that of power allegories and relation algebras, where relations and their algebras have been studied traditionally. In fact, the notions from power allegories and multirelations used here align very naturally with the structure studied. An exploration of the monadic approach and its comparison with the relation-algebraic one is nevertheless an interesting avenue for future work.

Last but not least, the approximation maps α, δ_o and δ_i are important for defining modal operators on multirelations, which arise in concurrent dynamic logic following Peleg [15] and Nerode and Wijesekera [12]. In fact, an algebraic formalisation of such operators in our multirelational language has been a starting point of this line of work. This is explored in the third part of this trilogy [4] and constitutes the main application of our results so far.

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Appendix A. Basis

As in [5], almost every operation in this article can be defined in terms of a basis of 6 operations that mix the relational and the multirelational language: the relational operations $-$, $\cap$, $/$ and the multirelational operations 1, $\Cup$, $*$. Here we extend the list from [5] with definitions of the operations from power allegories.

- $R \cup S = -(-R \cap -S)$
- $R - S = R \cap -S$
- $\emptyset = R \cap -R$
- $U = -\emptyset$
- $R{\uparrow} = R \Cup U$
- $\in = 1{\uparrow}$
- $Id = 1/1$
- $R^{\smile} = -(-Id/R)$
- $SR = -(-S/R^{\smile})$
- $\ni = \in^{\smile}$
- $R\backslash S = (S^{\smile}/R^{\smile})^{\smile}$
- $R \div S = (R\backslash S) \cap (R^{\smile}/S^{\smile})$
- $\Lambda(R) = R^{\smile} \div \in$
- $\mathcal{P}(R) = \Lambda(\ni R)$
- $R_{\mathcal{P}} = \mathcal{P}(R\ni)$
- $\mu = Id_{\mathcal{P}}$
- $\Omega = \in\backslash\in$
- $C = \in \div - \in$
- $\sim R = RC$
- $R \Cap S = \sim(\sim R \Cup \sim S)$
- $R{\downarrow} = X \Cap U$
- $R{\updownarrow} = R{\uparrow} \cap R{\downarrow}$
- $1_{\Cup} = 1 \Cap \sim 1$
- $1_{\Cap} = \sim 1_{\Cup}$
- $R^{\mathsf{d}} = -\sim R$
- $R \odot S = \sim(R * \sim S)$
- $R_* = (\Lambda(\ni 1) * 1^{\smile} R1)\mu$
- $\mathsf{A}_{\Cup} = U1$
- $\mathsf{A}_{\Cap} = \sim \mathsf{A}_{\Cup}$
- $\nu(R) = R - 1_{\Cup}$
- $\tau(R) = R \cap 1_{\Cup}$
- $\alpha(R) = R\ni$
- $\delta_i(R) = R{\downarrow} \cap \mathsf{A}_{\Cup}$
- $\delta_o(R) = 1R_{\mathcal{P}}$
- $\tilde{\delta_i}(R) = R{\uparrow} \cap \mathsf{A}_{\Cap}$
- $\tilde{\delta_o}(R) = \sim\delta_o(\sim R)$
- $dom(R) = Id \cap RR^{\smile}$
- $R \sqsubseteq_{\uparrow} S \Leftrightarrow S \subseteq R{\uparrow}$
- $R \sqsubseteq_{\downarrow} S \Leftrightarrow R \subseteq S{\downarrow}$
- $R \sqsubseteq_{\updownarrow} S \Leftrightarrow R \sqsubseteq_{\downarrow} S \wedge R \sqsubseteq_{\uparrow} S$

We could replace relational intersection $\cap$ with a multirelational intersection variant $\cap$ in the basis: relational $\cap$ is obtained by $R \cap S = \alpha(R1 \cap S1)$ which can be defined in terms of multirelational $\cap$ and the rest of the basis. Yet we do not know whether a multirelational $-$ could replace the relational variant. See the comments on the list in [5] for further information.